MLSS SYDNEY 2015
Models for Probability/Discrete Vectors with Bayesian Non-parametric Methods

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Acknowledgements

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Thanks to Lancelot James of HK UST for teaching me generalised IBP.

Legend

Color coding:

blue phrases: important terms and phrases;

green phrases: new terms;

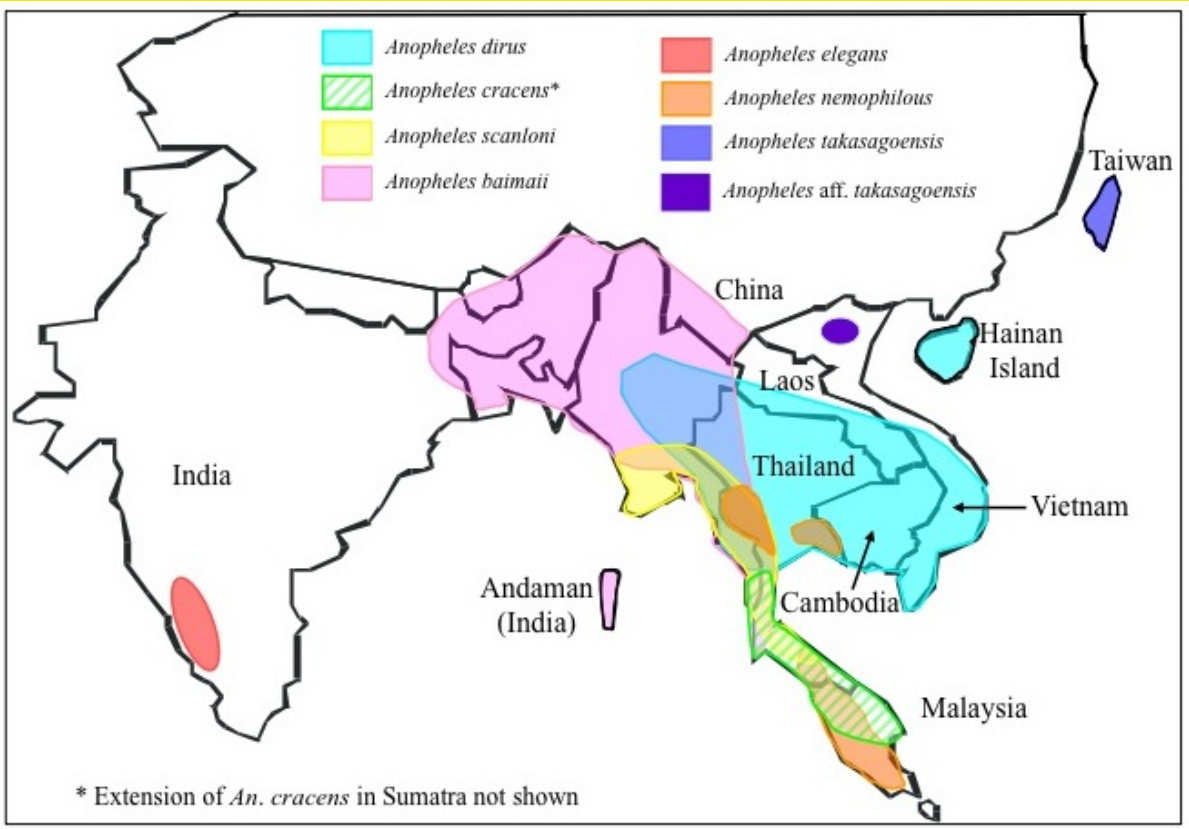
red phrases: important phrases with negative connotations.

Wikipedia is thin on the more esoteric parts of tutorial, but recommended content is marked so$^{\mathcal{W}}$.

Outline

Outline

How Many Species of Mosquitoes are There?



e.g. Given some measurement points about mosquitoes in Asia, how many species are there?

K=4? K=5? K=6 K=8?

How Many Words in the English Language are There?

> ... lastly, she pictured to herself how this same little sister of hers would, in the after-time, be herself a grown woman; and how she would keep, through all her riper years, the simple and loving heart of her childhood: and how she would gather about her other little children, and make their eyes bright and eager with many a strange tale, perhaps even with the dream of wonderland of long ago: ...

e.g. Given 10 gigabytes of English text, how many words are there in the English language?

K=1,235,791? K=1,719,765? K=2,983,548?

How Many are There?

How many species of mosquitoes are there?

- we expect there to be a finite number of species,
- we could use a Dirichlet of some fixed dimension K, and do model selection on K

$\longrightarrow$ Model with a finite mixture model of unknown dimension K.

How many words in the English language are there?

- This is a trick question.
- The *Complete Oxford English Dictionary* might attempt to define the language at some given point in time.
- The language keeps adding new words.
- The language is unbounded, it keeps growing.

$\longrightarrow$ Model with a countably infinite mixture model.

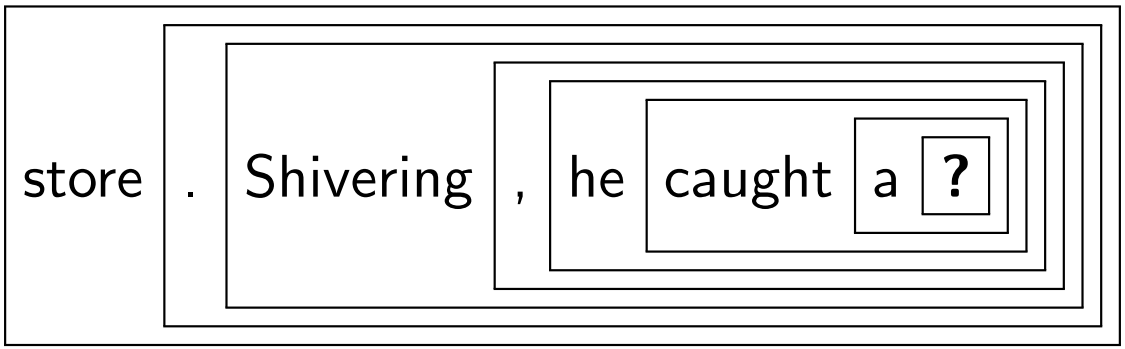
Probability Vectors

Problems in modern natural language processing and intelligent systems often have probability vectors for:

- the next word given $(n-1)$ previous,
- an author/conference/corporation to be linked to/from a webpage/patent/citation,
- part-of-speech of a word in context,
- hashtag in a tweet given the author.

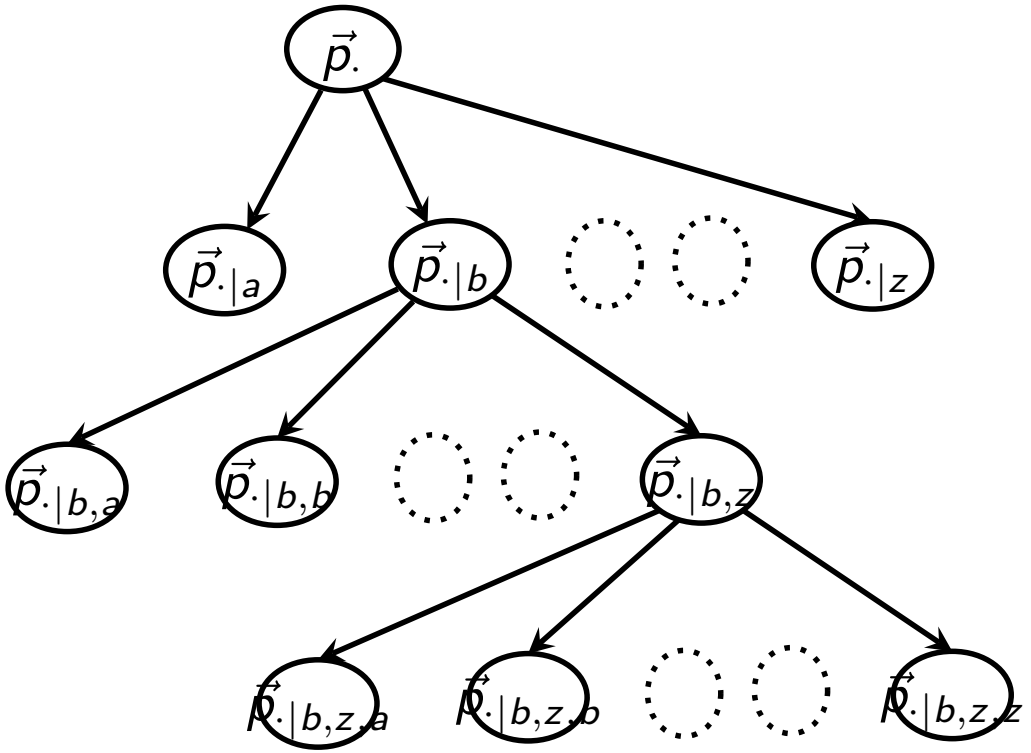
We need to work with distributions over probability vectors to model these sorts of phenomena well.

Bayesian Idea: Similar Context Means Similar Word



- Words in [a ?] should be like words in [?]
  - though no plural nouns
- Words in [caught a ?] should be like words in [a ?]
  - though a suitable object for "caught"
- Words in [he caught a ?] be *very like* words in [caught a ?]
  - "he" shouldn't change things much

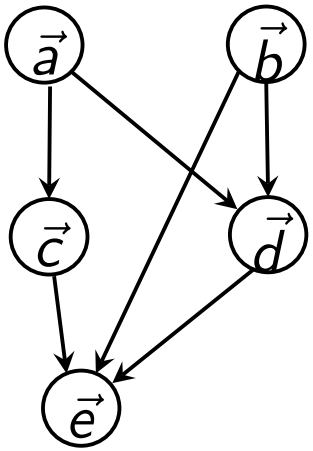
Network for Bayesian N-grams



$$\begin{aligned}
\mathcal{S} &= \text{symbol set, fixed or possibly countably infinite} \\
\vec{p}_\cdot &\sim \text{prior on prob. vectors (initial vocabulary)} \\
\vec{p}_{\cdot|x_1} &\sim \text{dist. on prob. vectors with mean } \vec{p}_\cdot \qquad \forall_{x_1 \in \mathcal{S}} \\
\vec{p}_{\cdot|x_1,x_2} &\sim \text{dist. on prob. vectors with mean } \vec{p}_{\cdot|x_1} \qquad \forall_{x_1,x_2 \in \mathcal{S}}
\end{aligned}$$

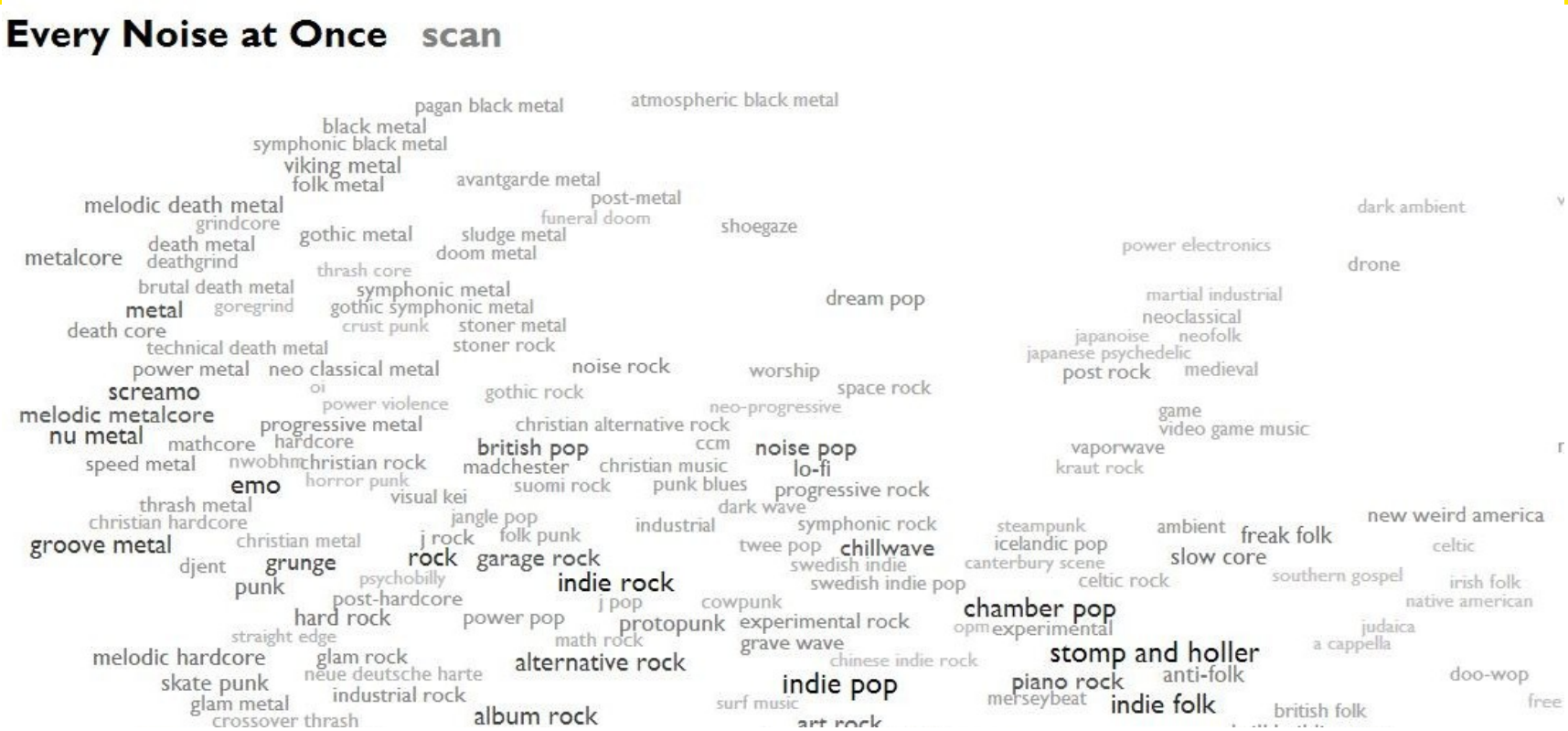
Networks/Hierarchies of Probability Vectors

- Early inference on Bayesian networks$^{\mathcal{W}}$ had categorical or Gaussian variables only.
- Subsequent research gradually extends the range of distributions.



- A large class of problems in NLP and machine learning require inference and learning on networks of probability vectors.
- Discrete non-parametric methods handle inference on networks of probabilty vectors efficiently. They do far more than just estimate "how many"!

Which Music Genres do You Listen to?



(see http://everynoise.com)

Music genres are constantly developing.

Which ones do you listen to?

What is the chance that a new genre is seen?

Which Ones are There and How Many?

Which music genres do you listen to?

- The available list is constantly expanding.
- You (may) have a small, fixed list you are aware of and actively listen to.

$\longrightarrow$ Model with a finite Boolean vector of unknown dimension K.

What Data is at Arnold Schwarzenegger's Freebase page?

- The list of entries is expanding: film performances, honorary degrees, profession, children, books, quotations, ...
- Which entries are there and how many of each?

$\longrightarrow$ Model with a finite count vector of unknown dimension K.

Discrete Matrix Data

Which ones? (Boolean matrix)

0	0	1	1	0	1	0
0	1	0	1	0	0	0
1	0	1	0	0	1	1
0	0	1	1	1	0	1
1	0	0	0	1	0	0

How many of each (count matrix)

0	0	1	3	0	2	0
0	4	0	2	0	0	0
1	0	3	0	0	4	2
0	0	4	1	1	0	3
2	0	0	0	1	0	0

Which structure? (*e.g.*,Boolean vector matrix)

0	0	(1,0,0)	(0,0,1)	0	(0,1,0)	0
0	(1,0,0)	0	(0,1,0)	0	0	0
(1,0,0)	0	(0,0,1)	0	0	(1,0,0)	(0,1,0)
0	0	(1,0,0)	(1,0,0)	(1,0,0)	0	(0,0,1)
(0,1,0)	0	0	0	(1,0,0)	0	0

Seductive Semantics: Example Topics

Sets of topic words created from the New York Times 1985-2005 news collection.
i.e., simplifying a 800,000 by 15,000 matrix using 1000^2 diagonal core

- career,born,grew,degree,earned,graduated,became,studied,graduate
- mother,daughter,son,husband,family,father,parents,married,sister
- artillery,shells,tanks,mortars,gunships,rockets,firing,tank
- clues,investigation,forensic,inquiry,leads,motive,investigator,mystery
- freedom,tyranny,courage,america,deserve,prevail,evil,bless,enemies
- viewers,cbs,abc,cable,broadcasting,channel,nbc,broadcast,fox,cnn
- anthrax,spores,mail,postal,envelope,powder,letters,daschle,mailed

Topic models yield high-fidelity semantic associations!

Seductive Semantics: Example Topics II

1000 sparse topics built on text content from NIPS 1987-2012.
i.e., simplifying a 50,000 by 5,000 matrix using 1000^2 diagonal core

100: collapsed, variational, mixed-membership, mean-field, inference, hopls, e-step, probit, m-step, topic-lda, non-conjugate, teh, posteriors, latent, analyzers, marginalised, posterior, summaries

576: priors, prior, jeffreys, posteriori, non-informative, posterior, posteriors, peaked, priori, beta, improper, noninformative, bp-lr, ald, favors, log-posterior, specification

645: bayesian, priors, prior, bayes, posterior, non-bayesian, frequentist, posteriors, conjugate, vague, integrating, conjugacy, precisions, averaging, likelihoods, bma

791: posterior, inference, posteriors, gpc, unscented, approximations, laplace, approximate, mixture-of-gaussians, intractable, exact, inferential, factorizes, approximation, mcmc, unimodal,

Bayesian Inference

Bayesian inference$^{\mathcal{W}}$ is particularly suited for intelligent systems in the context of the previous requirments:

- Bayesian model combination$^{\mathcal{W}}$ and Bayes factors$^{\mathcal{W}}$ for model selection$^{\mathcal{W}}$ can be used;
- marginal likelihood$^{\mathcal{W}}$, a.k.a. the evidence for efficient estimation;
- collapsed Gibbs samplers$^{\mathcal{W}}$, a.k.a. Rao-Blackwellised samplers, for Monte-Carlo Markov chain$^{\mathcal{W}}$ (MCMC) estimation;
- also blocked Gibbs samplers$^{\mathcal{W}}$.

NB. Wikipedia coverage of Bayesian non-parametrics is poor.

But can the non-parametric inference be made practical?

ASIDE: Aspects, Ratings and Sentiments

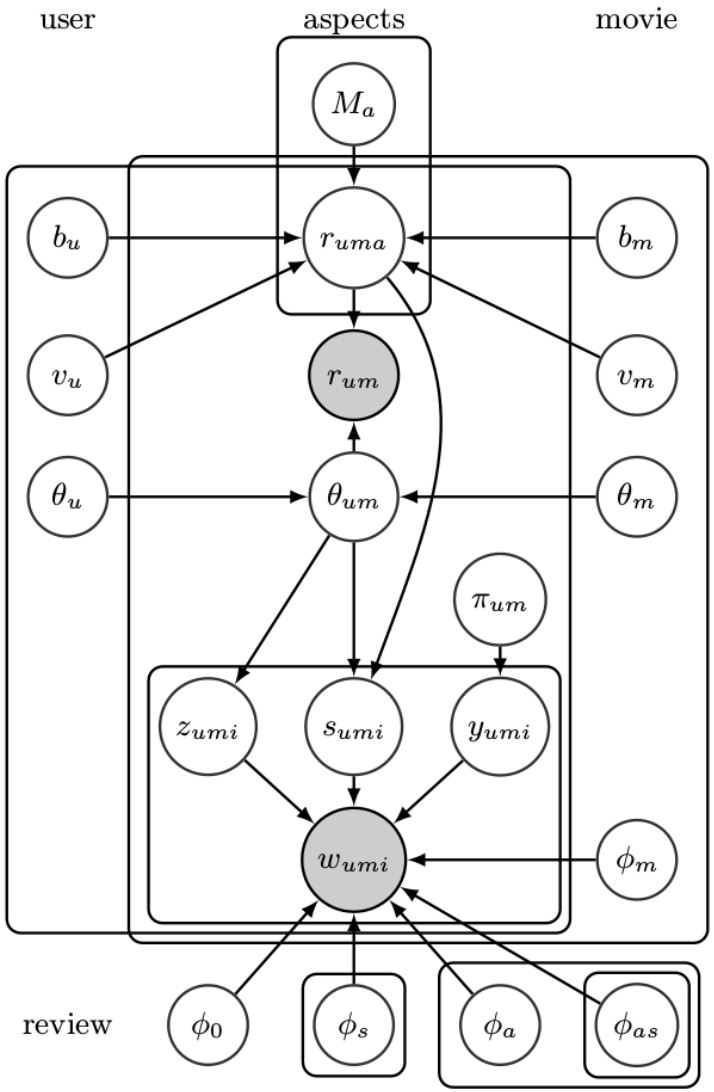


Figure 1: Factorized rating and review model.

"Jointly Modeling Aspects, Ratings and Sentiments for Movie Recommendation (JMARS)," Diao, Qiu, Wu, Smola, Jiang and Wang, KDD 2014.

State of the art sentiment model.

Typical methods currently lack probability vector hierarchies.

Chinese Restaurants and Breaking Sticks

Standard machine learning methods for dealing with probability vectors are based on:

- Dirichlet Processes (DPs) and Pitman-Yor processes (PYPs).
- stick-breaking versions for infinite probability vectors, and
- Chinese restaurant process (CRP) versions for distributions on probability vectors.

Historical Context of DPs and PYPs

1990s: Pitman and colleagues in mathematical statistics develop statistical theory of partitions, Pitman-Yor process, *etc.*

2006: Teh develops hierarchical n-gram models using PYs.

2006: Teh, Jordan, Beal and Blei develop hierarchical Dirichlet processes, *e.g.* applied to LDA.

2006-2011: Chinese restaurant processes (CRPs) go wild!

- Chinese restaurant franchise,
- multi-floor Chinese restaurant process (Wood and Teh, 2009),
- huge range of problems in ML and NLP especially. *etc.*

2010: Johnson develops Adaptor Grammars

Opened up whole field of application for non-parametrics.

Chinese Restaurants and Breaking Sticks in Hierarchical Models

Wray's opinions:

- With hierarchical models, the CRP tends to hide aspects of the standard Bayesian framework: the actual posterior, the underlying model.
- With hierarchical models, the CRP requires considerable dynamic memory for use on larger problems.
- With hierarchical models, the stick-breaking model seems to interact poorly with variational algorithms.
- The stick-breaking model involves an inherent order on the clusters that slightly alters the posterior.

Goals of the Tutorial

- We'll see how to address the problems:
  - distributions on probability vectors,
  - countably infinite mixture models,
  - infinite discrete feature vectors.
- We'll use the Dirichlet Process (DP), the Pitman-Yor Process (PYP), and a generalisation of Indian Buffet Process (IBP)
- We'll see how to develop complex models and samplers using these.
- The methods are ideal for tasks like sharing, inheritance and arbitrarily complex models, all of which are well suited for Bayesian methods.
- The analysis will be done in the context of the standard Bayesian practice.

Outline

Discrete Distributions

Name	Domain	$p(x\|\cdots)$
Bernoulli(ρ)	$x \in \{0,1\}$	$\rho^x(1-\rho)^{1-x}$
categorical$(K,\vec{\lambda})$	$x \in \{1,...,K\}$	λ_x
Poisson(λ)	$x \in \mathcal{N} = \{0,1,...,\infty\}$	$\frac{1}{x!}\lambda^x e^{-\lambda}$
multinomial$(K,N,\vec{\lambda})$	$\vec{n} \in \mathcal{N}^K$ s.t. $\sum_{k=1}^K n_k = N$	$\binom{N}{\vec{n}} \prod_{k=1}^K \lambda_k^{n_k}$
negative-binomial(λ,ρ)	$x \in \mathcal{N}$	$\frac{1}{x!}(\lambda)_x \rho^x(1-\rho)^\lambda$

$\rho \in (0,1)$, $\lambda \in \mathcal{R}^+$, $\lambda_k \in \mathcal{R}^+$ and $\sum_k \lambda_k = 1$

- A multinomial is a (unordered) set of categoricals.
- A multinomial also comes from normalising Poissons.
- A negative binomial comes from marginalising out the λ of a Poisson, giving it a Gamma distribution.

Conjugate Distributions

Name	Domain	$p(\lambda \mid \cdots)$
Beta(α, β)	$\lambda \in (0, 1)$	$\frac{1}{\text{Beta}(\alpha,\beta)} \lambda^{\alpha-1}(1-\lambda)^{\beta-1}$
Dirichlet$(\vec{\alpha})$	$\lambda_k \in (0, 1)$ s.t. $\sum_{k=1}^{K} \lambda_k = 1$	$\frac{1}{\text{Beta}_K(\vec{\alpha})} \prod_{k=1}^{K} \lambda_k^{\alpha_k - 1}$
Gamma(α, β)	$\lambda \in (0, \infty)$	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1} e^{-\beta\lambda}$

$\alpha, \beta > 0$, $\alpha_k > 0$

- Beta is a 2-D case of the K-dimensional Dirichlet
- A Dirichlet comes from normalising Gamma's with same scale β.

ASIDE: Divisability of Distributions

Data can be split and merged across dimensions:

Bernoulli; $x_1 \sim \text{Bernoulli}(\lambda_1)$ and $x_2 \sim \text{Bernoulli}(\lambda_2)$ and x_1, x_2 mutually exclusive then $(x_1 + x_2) \sim \text{Bernoulli}(\lambda_1 + \lambda_2)$.

Poisson: $x_1 \sim \text{Poisson}(\lambda_1)$ and $x_2 \sim \text{Poisson}(\lambda_2)$ then $(x_1 + x_2) \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

negative-binomial: $x_1 \sim \text{NB}(\lambda_1, p)$ and $x_2 \sim \text{NB}(\lambda_2, p)$ then $(x_1 + x_2) \sim \text{NB}(\lambda_1 + \lambda_2, p)$.

Gamma: $\lambda_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $\lambda_2 \sim \text{Gamma}(\alpha_2, \beta)$ then $(\lambda_1 + \lambda_2) \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$.

Normalising Distributions

- A multinomial comes from normalising Poissons.
  Let $X = \sum_{k=1}^{K} x_k$, then:

  $x_k \sim Poisson(\lambda_k)$ *for* $k \in \{1, ..., K\}$ *is equivalent to*

  $X \sim Poisson\left(\sum_{k=1}^{K} \lambda_k\right)$ *and* $\vec{x} \sim multinomial\left(K, X, \vec{\lambda}\right)$

- A Dirichlet comes from normalising Gammas with same scale β.
  Let $\lambda_0 = \sum_{k=1}^{K} \lambda_k$, then:

  $\lambda_k \sim Gamma(\alpha_k, \beta)$ *for* $k \in \{1, ..., K\}$ *is equivalent to*

  $\lambda_0 \sim Gamma\left(\sum_{k=1}^{K} \alpha_k, \beta\right)$ *and* $\frac{1}{\lambda_0}\vec{\lambda} \sim Dirichlet_K\left(\vec{\alpha}\right)$

Part of these results comes from divisability.

Dirichlet Distribution

Definition of Dirichlet distribution

The Dirichlet distribution$^{\mathcal{W}}$ is used to sample finite probability vectors.

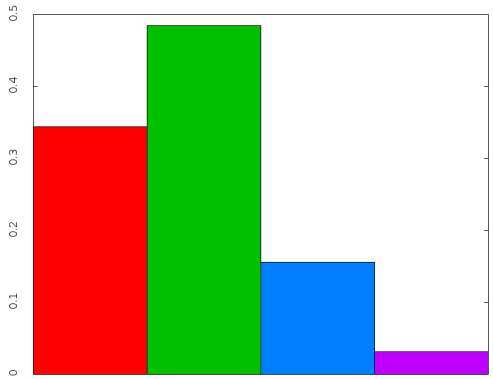
$$\vec{p} \sim \text{Dirichlet}_K(\vec{\alpha})$$

where $\alpha_0 > 0$ and $\vec{\mu}$ is a positive K-dimensional probability vector; alternatively $\vec{\alpha}$ is a positive K-dimensional vector.

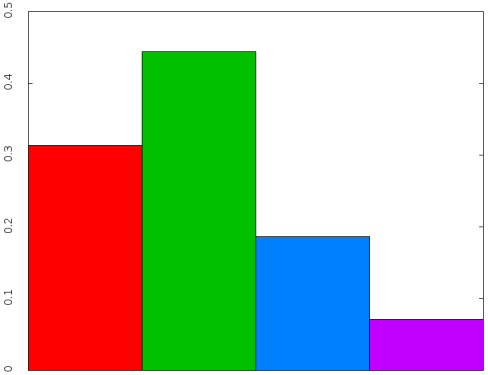
- alternate form $\text{Dirichlet}_K(\alpha_0, \vec{\mu})$ comparable to the circular multivariate Gaussian $\vec{x} \sim \text{Gaussian}_K(\sigma^2, \vec{\mu})$ (mean, concentration, *etc.*),
- said to be a conjugate prior$^{\mathcal{W}}$ for the multinomial distribution, *i.e.*, makes math easy.

4-D Dirichlet samples

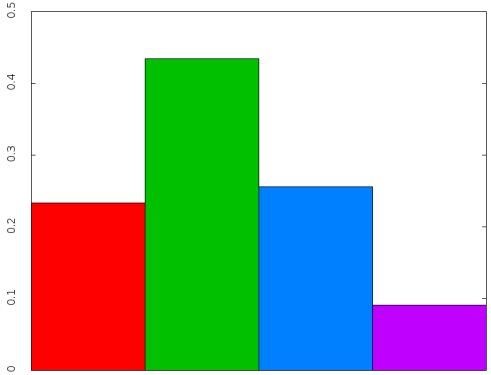
$\vec{p}_0$



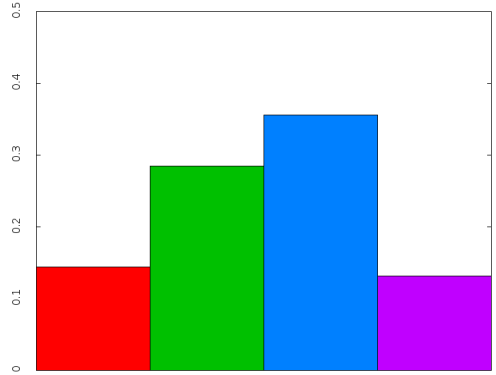
$\vec{p}_1 \sim \text{Dirichlet}_4(500, \vec{p}_0)$



$\vec{p}_2 \sim \text{Dirichlet}_4(5, \vec{p}_0)$

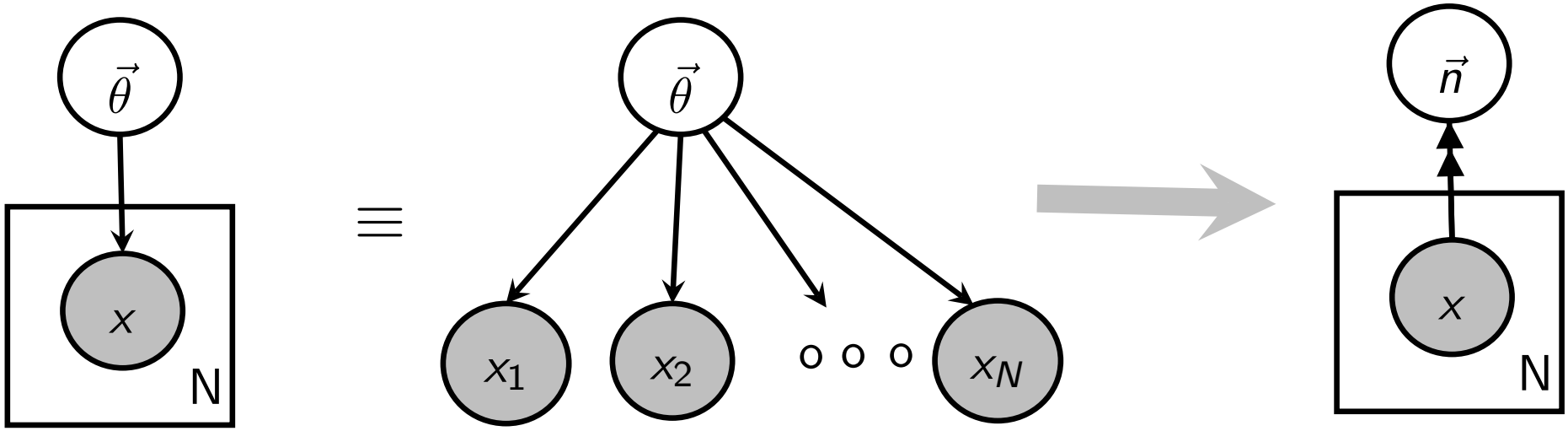


$\vec{p}_3 \sim \text{Dirichlet}_4(0.5, \vec{p}_0)$



Outline

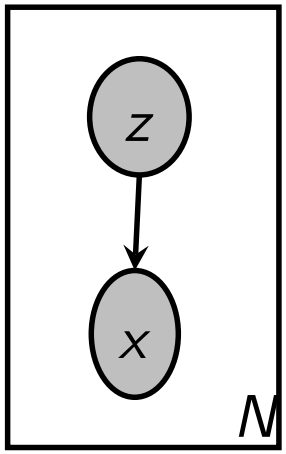
Reading a Graphical Model with Plates



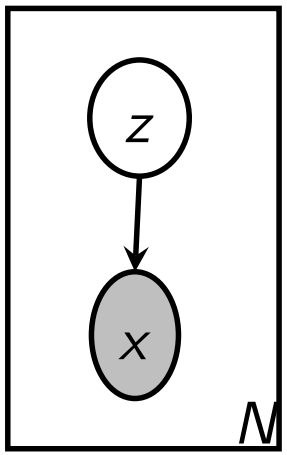
arcs	=	"depends on"
double headed arcs	=	"deterministically computed from"
shaded nodes	=	"supplied variable/data"
unshaded nodes	=	"unknown variable/data"
boxes	=	"replication"

Models in Graphical Form

Supervised learning or Prediction model

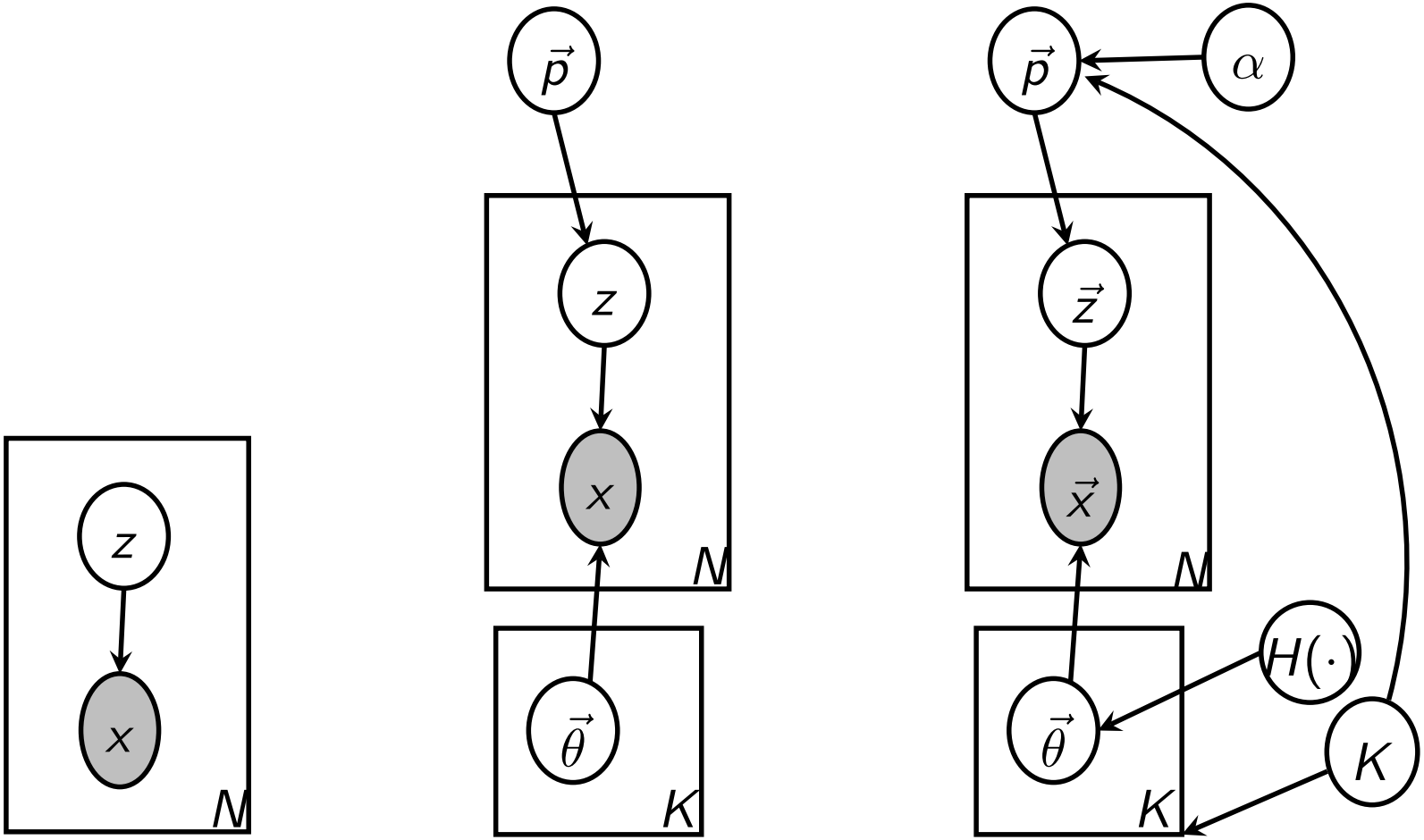


Clustering or Mixture model



Mixture Models in Graphical Form

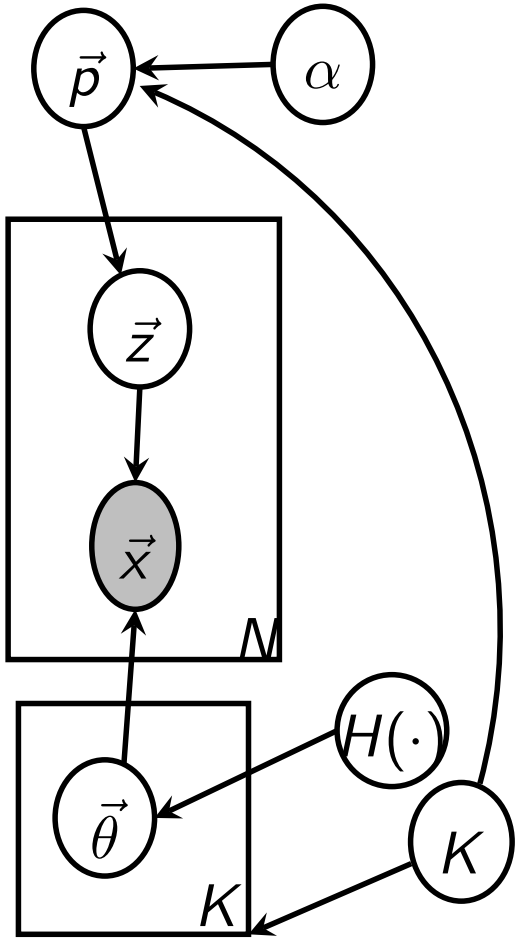
Building up the parts:



The Classic Discrete Mixture Model

Data is a mixture of unknown dimension K. Base distribution $H(\cdot)$ generates the distribution for each cluster/component $\vec{\theta}_k$.

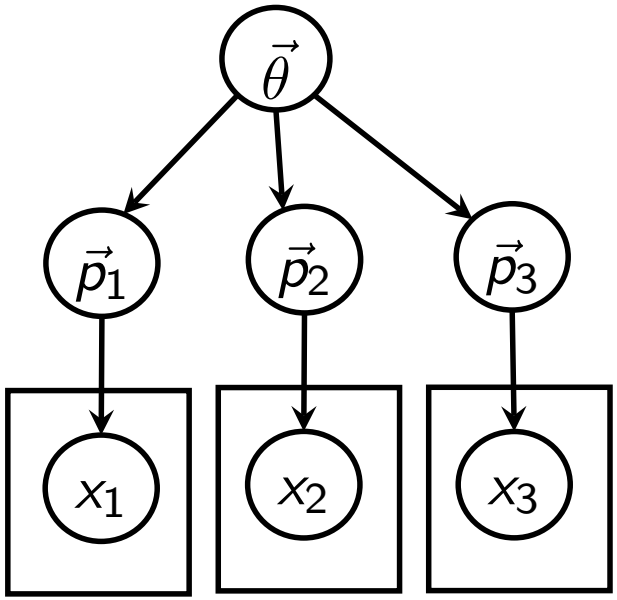
$$
\begin{aligned}
K &\sim G(\cdot) \\
\vec{p} &\sim \text{Dirichlet}_K\left(\frac{\alpha}{K}\vec{1}\right) \\
\vec{\theta}_k &\sim H(\cdot) && \forall_{k=1,\ldots,K} \\
z_n &\sim \vec{p} && \forall_{n=1,\ldots,N} \\
x_n &\sim \vec{\theta}_{z_n} && \forall_{n=1,\ldots,N}
\end{aligned}
$$



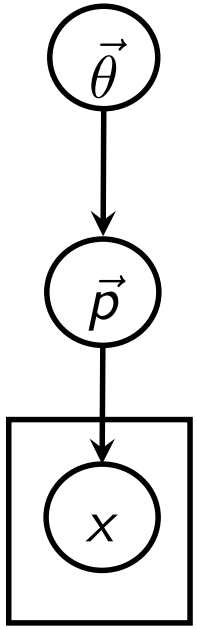
Outline

Dirichlet-Multinomial Motivation

A useful inference component is the Dirichlet-Multinomial. Begin by adding multinomials off the samples from a Dirichlet.



$$\vec{p}_l \sim \text{Dirichlet}\left(\alpha, \vec{\theta}\right) \quad \forall_l$$
$$x_{l,n} \sim \text{Discrete}(\vec{p}_l) \quad \forall_{l,n}$$

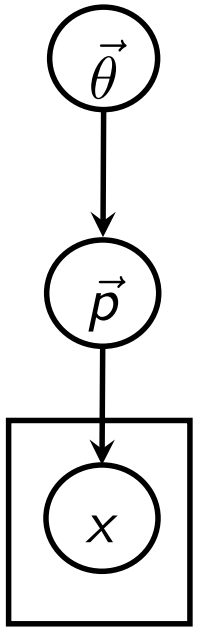


$$\vec{p} \sim \text{Dirichlet}\left(\alpha, \vec{\theta}\right)$$
$$x_n \sim \text{Discrete}(\vec{p}) \quad \forall_n$$

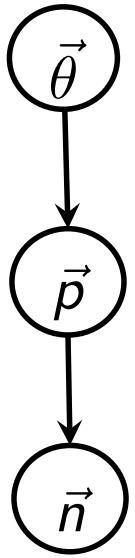
We will analyse the simplest case on the right.

The Dirichlet-Multinomial

First convert the categorical data into a set of counts.

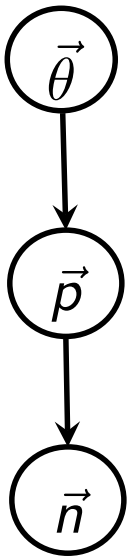


$$\begin{aligned} \vec{p} &\sim \text{Dirichlet}\left(\alpha, \vec{\theta}\right) \\ x_n &\sim \text{Discrete}(\vec{p}) \quad \forall_n \end{aligned}$$



$$\begin{aligned} \vec{p} &\sim \text{Dirichlet}\left(\alpha, \vec{\theta}\right) \\ \vec{n} &\sim \text{Multinomial}\left(\vec{p}, N\right) \end{aligned}$$

The Dirichlet-Multinomial, cont



$$\vec{p} \sim \text{Dirichlet}\left(\alpha, \vec{\theta}\right) \quad \forall_k$$

$$\vec{n} \sim \text{Multinomial}\left(\vec{p}, N\right)$$

$$p\left(\vec{p}, \vec{n} \middle| \vec{\theta}, ...\right)$$

$$= \frac{1}{\text{Beta}(\alpha\vec{\theta})} \left(\prod_k p_k^{\alpha\theta_k - 1}\right) \binom{N}{\vec{n}} \prod_k p_k^{n_k}$$

Integrate out (or eliminate/marginalise) $\vec{p}$:

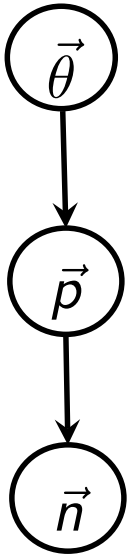
$$p\left(\vec{n} \middle| \vec{\theta}, ...\right)$$

$$= \frac{1}{\text{Beta}(\alpha\vec{\theta})} \binom{N}{\vec{n}} \int_{\text{simplex}} \prod_k p_k^{n_k + \alpha\theta_k - 1} \text{d}\vec{p}$$

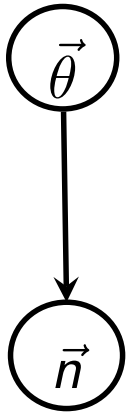
$$= \binom{N}{\vec{n}} \frac{\text{Beta}(\vec{n} + \alpha\vec{\theta})}{\text{Beta}(\alpha\vec{\theta})}$$

The Dirichlet-Multinomial, cont

The distribution with $\vec{p}$ marginalised out is given on right:



$$\vec{p} \sim \text{Dirichlet}\left(\alpha, \vec{\theta}\right)$$
$$\vec{n} \sim \text{Multinomial}\left(\vec{p}, N\right)$$



$$\vec{n} \sim \text{MultDir}\left(\alpha, \vec{\theta}, N\right)$$
$$\text{where } \sum_k n_k = N$$
$$p\left(\vec{n} \middle| N, \text{MultDir}, \alpha, \vec{\theta}\right) = \binom{N}{\vec{n}} \frac{\text{Beta}(\vec{n} + \alpha\vec{\theta})}{\text{Beta}(\alpha\vec{\theta})}$$

The Dirichlet-Multinomial, cont

Definition of Dirichlet-Multinomial

Given a concentration parameter α, a probability vector $\vec{\theta}$ of dimension K, and a count N, the Dirichlet-multinomial distribution creates count vector samples $\vec{n}$ of dimension K. Now $\vec{n} \sim \text{MultDir}\left(\alpha, \vec{\theta}, N\right)$ denotes

$$p\left(\vec{n}\middle| N, \text{MultDir}, \alpha, \vec{\theta}\right) = \binom{N}{\vec{n}} \frac{\text{Beta}\left(\vec{n} + \alpha\vec{\theta}\right)}{\text{Beta}\left(\alpha\vec{\theta}\right)}$$

where $\sum_{k=1}^{K} n_k = N$ and $\text{Beta}(\cdot)$ is the normalising function for the Dirichlet distribution.

This probability is also the evidence (probability of data with parameters marginalised out) for a Dirichlet distribution.

A Hierarchical Dirichlet-Multinomial Component?

Consider the functional form of the MultDir.

$$
\begin{aligned}
p\left(\vec{n}\middle| N, \text{MultDir}, \alpha, \vec{\theta}\right) &= \binom{N}{\vec{n}} \frac{\text{Beta}\left(\alpha\vec{\theta} + \vec{n}\right)}{\text{Beta}\left(\alpha\vec{\theta}\right)} \\
&= \binom{N}{\vec{n}} \frac{1}{(\alpha)_N} \prod_{k=1}^{K} (\alpha\theta_k)(\alpha\theta_k + 1)\cdots(\alpha\theta_k + n_k - 1) \\
&= \binom{N}{\vec{n}} \frac{1}{(\alpha)_N} \prod_{k=1}^{K} (\alpha\theta_k)_{n_k}
\end{aligned}
$$

where $(x)_n = x(x+1)...(x+n-1)$ is the rising factorial.

This is a complex polynomial we cannot deal with in a hierarchical model.

Outline

Topic Models of Text

Original news article:

Despite their separation, Charles and Diana stayed close to their boys William and Harry. Here, they accompany the boys for 13-year-old William's first day school at Eton College on Sept. 6, 1995, with housemaster Dr. Andrew Gayley looking on.

Bag of words:

13 1995 accompany and(2) andrew at boys(2) charles close college day despite diana dr eton first for gayley harry here housemaster looking old on on school separation sept stayed the their(2) they to william(2) with year

We'll approximate the bag with a linear mixture of text topics as probability vectors.

Components:

Words (probabilities not shown)	Human label
Prince, Queen, Elizabeth, title, son, ...	Royalty
school, student, college, education, year, ...	School
John, David, Michael, Scott, Paul, ...	Names
and, or, to , from, with, in, out, ...	Function

Matrix Approximation View

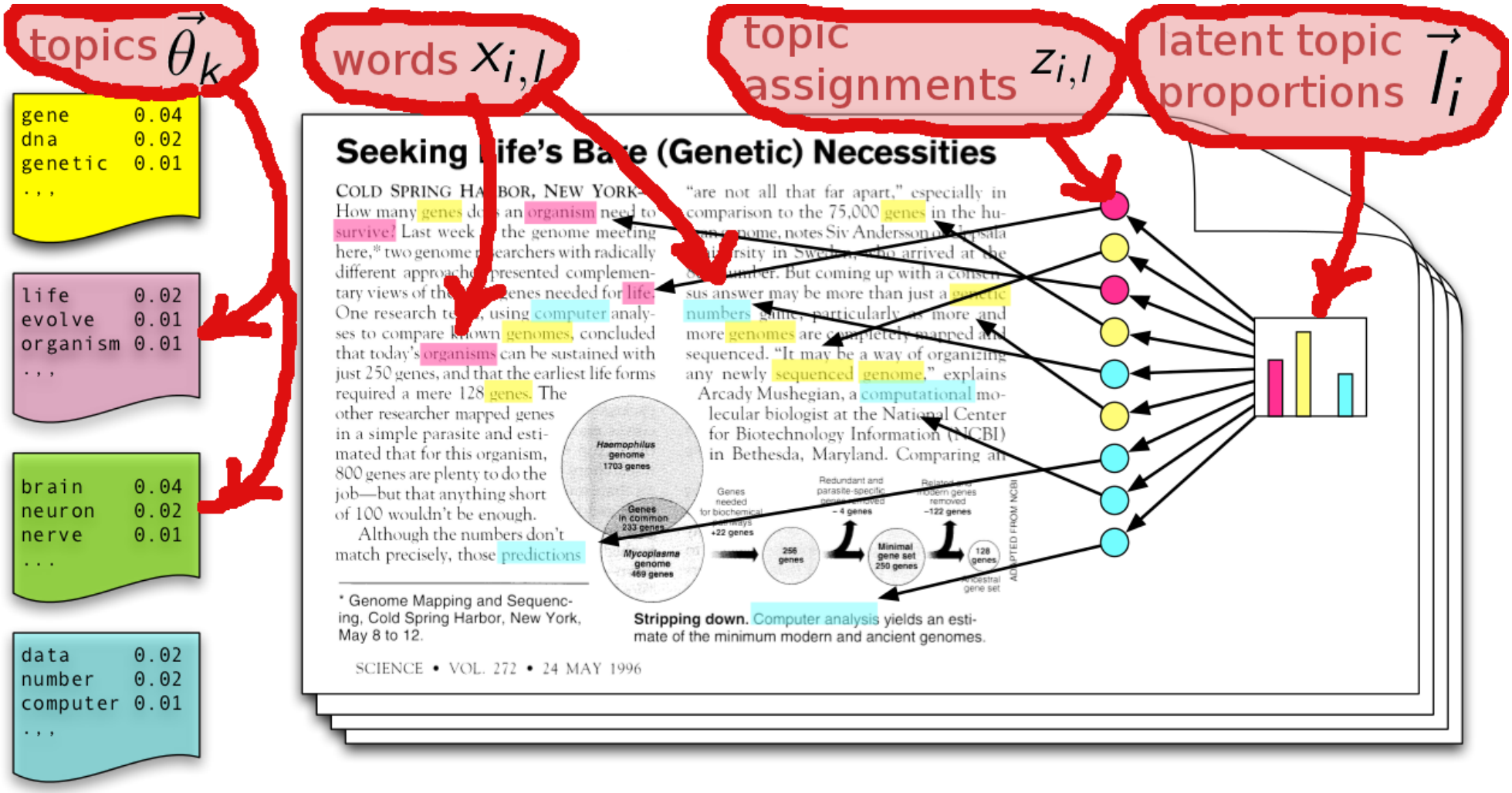
$$\boldsymbol{W} \simeq \boldsymbol{L} * \boldsymbol{\Theta}^T$$

$$\underbrace{\begin{pmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,J} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,J} \\ \vdots & \vdots & \ddots & \vdots \\ w_{I,1} & w_{I,2} & \cdots & w_{I,J} \end{pmatrix}}_{J \text{ words}} \Big\} \text{I documents} \simeq \underbrace{\begin{pmatrix} l_{1,1} & \cdots & l_{1,K} \\ l_{2,1} & \cdots & l_{2,K} \\ \vdots & \ddots & \vdots \\ l_{I,1} & \cdots & l_{I,K} \end{pmatrix}}_{K \text{ components}} * \underbrace{\begin{pmatrix} \theta_{1,1} & \theta_{2,1} & \cdots & \theta_{J,1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{1,K} & \theta_{2,K} & \cdots & \theta_{J,K} \end{pmatrix}}_{J \text{ words}} \Big\} \text{K components}$$

Different variants:

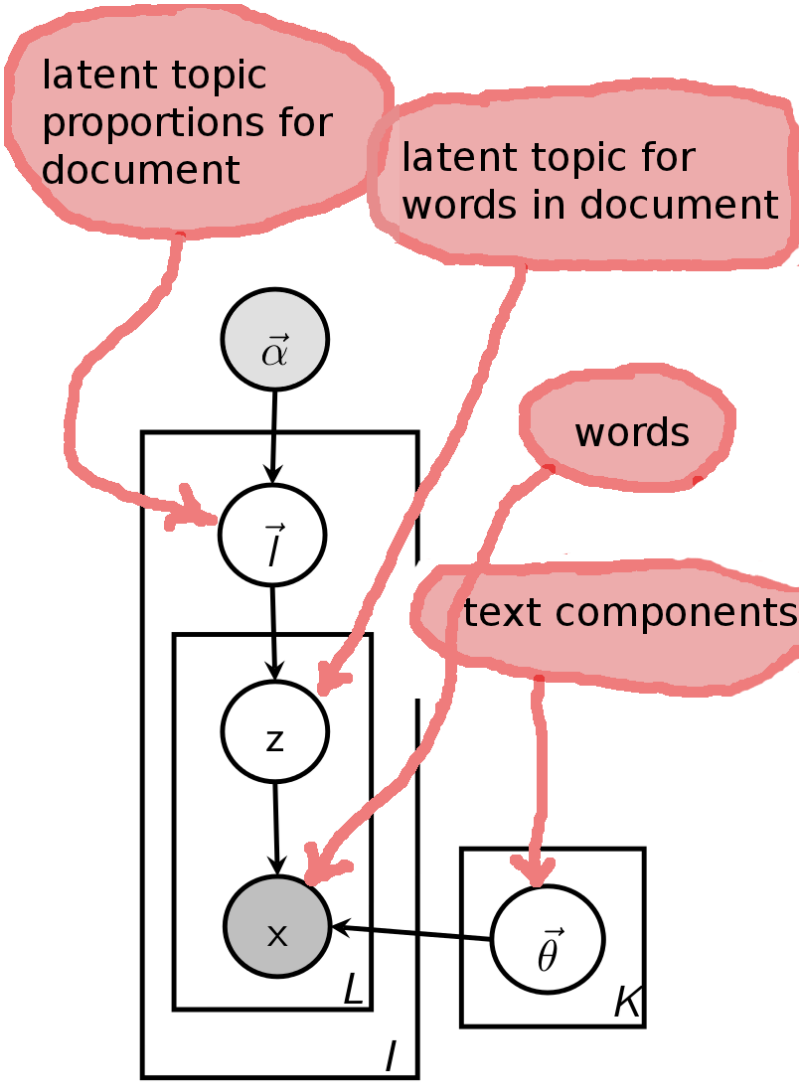
Data $\boldsymbol{W}$	Components $\boldsymbol{L}$	Error	Models
real valued	unconstrained	least squares	PCA and LSA
non-negative	non-negative	least squares	codebooks, NMF
non-neg integer	non-negative	cross-entropy	topic modelling, NMF
real valued	independent	small	ICA

Clustering Words in Documents View



Blei's MLSS 2009 talk, with annotation by Wray.

LDA Topic Model



$$\vec{\theta}_k \sim \text{Dirichlet}_V(\vec{\gamma}) \quad \forall_{k=1}^{K} ,$$
$$l_i \sim \text{Dirichlet}_K(\vec{\alpha}) \quad \forall_{i=1}^{I} ,$$
$$z_{i,l} \sim \text{Discrete}\left(\vec{l}_i\right) \quad \forall_{i=1}^{I} \forall_{l=1}^{L_i} ,$$
$$x_{i,l} \sim \text{Discrete}\left(\vec{\theta}_{z_{i,l}}\right) \quad \forall_{i=1}^{I} \forall_{l=1}^{L_i} .$$

where

$$K := \#\text{ topics},$$
$$V := \#\text{ words},$$
$$I := \#\text{ documents},$$
$$L_i := \#\text{ words in doc } i$$

Collapsed LDA Inference

$$
\begin{array}{rcll}
\vec{\theta}_k & \sim & \text{Dirichlet}_V\left(\vec{\gamma}\right) & \forall_{k=1}^{K} \\
\vec{l}_i & \sim & \text{Dirichlet}_K\left(\vec{\alpha}\right) & \forall_{i=1}^{I} \\
z_{i,l} & \sim & \text{Discrete}\left(\vec{l}_i\right) & \forall_{i=1}^{I} \forall_{l=1}^{L_i} \\
x_{i,l} & \sim & \text{Discrete}\left(\vec{\theta}_{z_{i,l}}\right) & \forall_{i=1}^{I} \forall_{l=1}^{L_i}
\end{array}
$$

where

$$
\begin{array}{rcl}
K & := & \#\text{ topics}, \\
V & := & \#\text{ words}, \\
I & := & \#\text{ documents}, \\
L_i & := & \#\text{ words in doc } i
\end{array}
$$

The LDA posterior is collapsed by marginalising out $\forall_i \vec{l}_i$ and $\forall_i \vec{\theta}_i$:

$$
\prod_{i=1}^{I} \frac{\text{Beta}_K\left(\vec{\alpha} + \vec{m}_i\right)}{\text{Beta}_K\left(\vec{\alpha}\right)} \prod_{k=1}^{K} \frac{\text{Beta}_V\left(\vec{\gamma} + \vec{n}_k\right)}{\text{Beta}_V\left(\vec{\gamma}\right)}
$$

where

$$
\begin{array}{rcl}
\vec{m}_i & := & \dim(K) \text{ data counts of topics for doc } i, \\
\vec{n}_k & := & \dim(V) \text{ data counts of words for topic } k.
\end{array}
$$

See the Dirichlet-multinomials!

So people have trouble making LDA hierarchical!

Outline

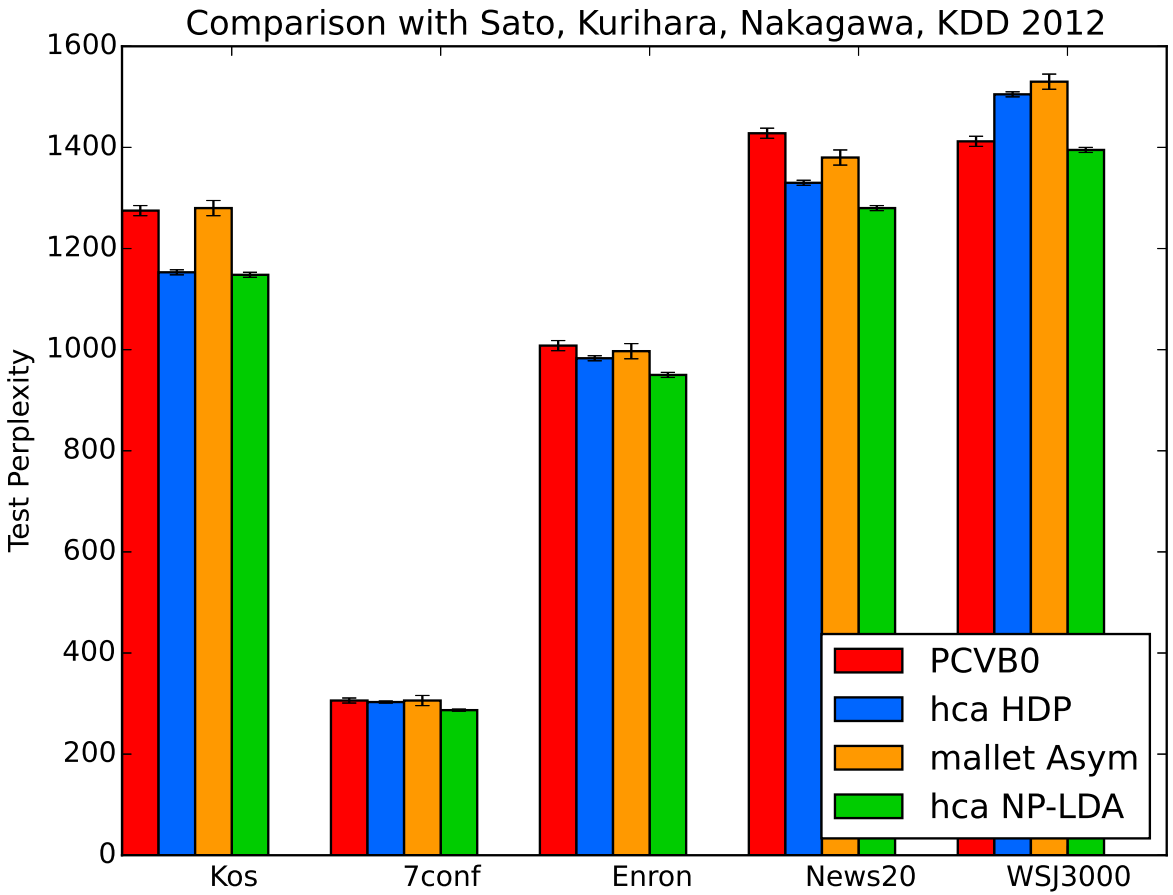
Evaluation of Topic Models

David Lewis (Aug 2014) "topic models are like a Rorschach inkblot test" (not his exact words .... but the same idea)

Perplexity:
- measure of test set likelihood;
- equal to effective size of vocabulary;
- we use "document completion," see Wallach, Murray, Salakhutdinov, and Mimno, 2009;
- however it is not a bonafide evaluation task

Comparisons: compare `hca` implementation of HDP-LDA and a non-parametric NP-LDA to published methods.

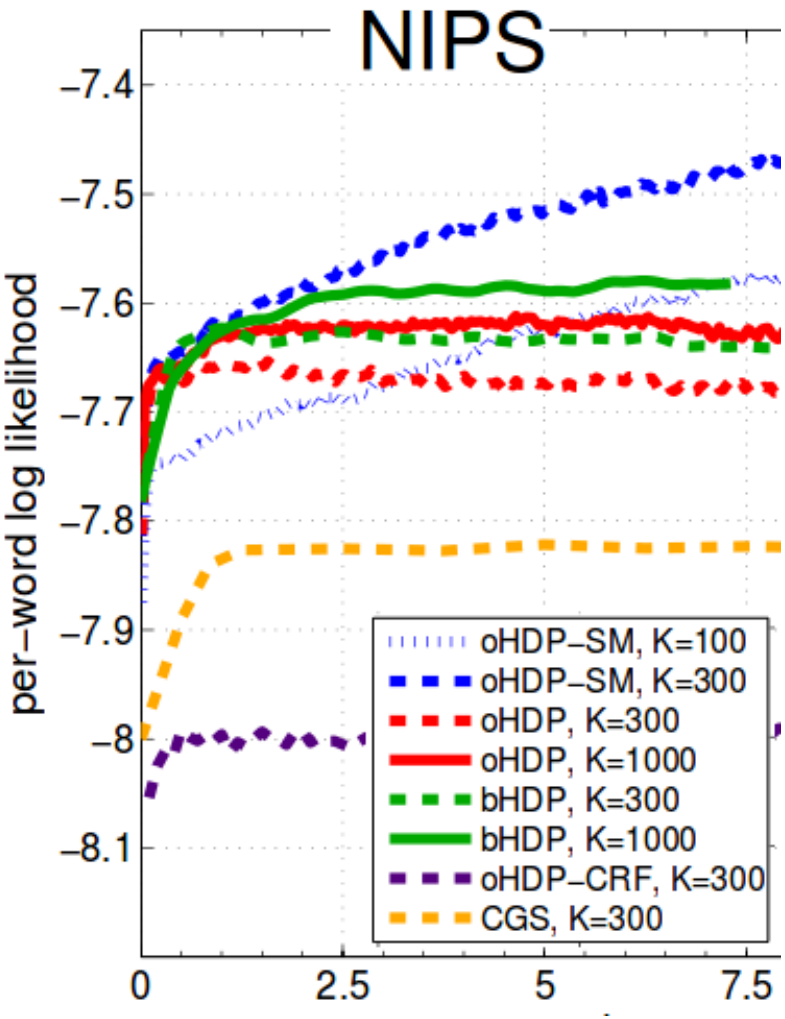
Comparison to PCVB0 and Mallet

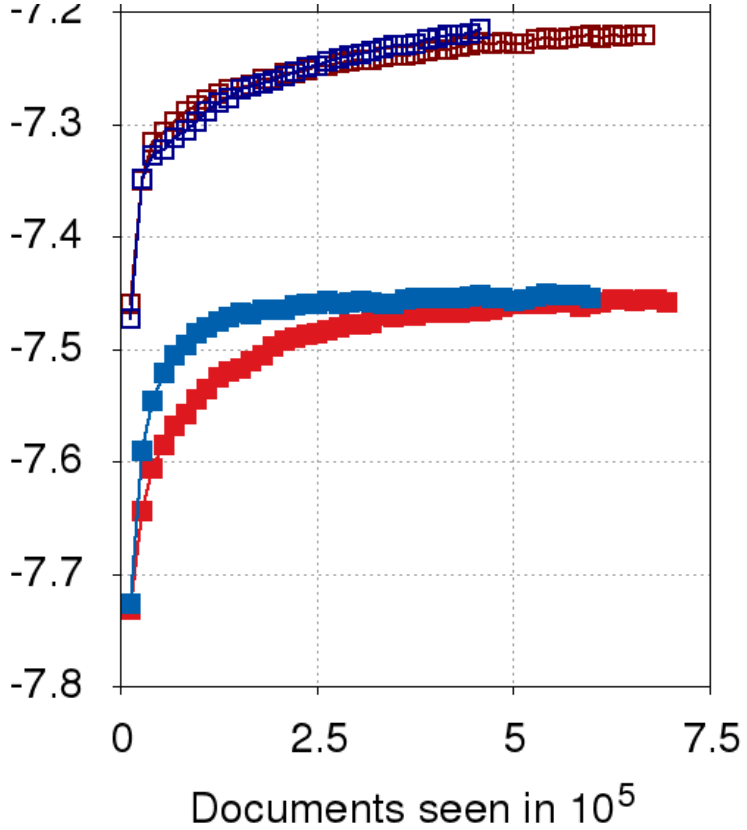


Protocol is train on 80% of all documents then using trained topic probs get predictive probabilities on remaining 20%, and replicate 5 times.

- Data contributed by Sato. Protocol by Sato *et al.*
- PCVB0 is a refined variational HDP-LDA by Sato, Kurihara, Nakagawa KDD 2012.
- Mallet (asymmetric-symmetric) is truncated HDP-LDA

Comparison to Bryant+Sudderth (2012) on NIPS data





(hca versions)

Comparison to FTM and LIDA

FTM and LIDA use IBP models to select words/topics within LDA.
Archambeau, Lakshminarayanan, and Bouchard, *Trans IEEE PAMI* 2014.

Data	KOS	NIPS
FTM (1-par)	7.262±0.007	6.901±0.005
FTM (3-par)	7.266±0.009	6.883±0.008
LIDA	7.257±0.010	6.795±0.007
hca HPD-LDA	7.253±0.003	6.792±0.002
time	3 min	22 min
hca NP-LDA	7.156±0.003	6.722±0.003

- KOS data contributed by Sato (D=3430, V=6906). NIPS data from UCI (D=1500, V=12419).
- Protocol same as with PCVB0 but a 50-50 split. Figures are log perplexity. Using 300 cycles.

- Better implementation of HDP-LDA now similar to LIDA.
- But LIDA still substantially better than LDA so we need to consider combining the technique with NP-LDA.

Comparisons on Non-parametric Topic Models

- Our technique (presented later, block table indicator sampling) substantially out-performs other techniques in perplexity and single CPU computational cost.
- Moderately easily parallelised for 4 to 8-cores CPUs.

Outline

Gibbs Sampling

- The simplest form of Monte Carlo Mark Chain sampling.
- Theory justified using Metropolis-Hasting theory.
- Sequence through each dimension/variable in turn. To sample a chain $\vec{\theta}^{(1)}, \vec{\theta}^{(2)}, \vec{\theta}^{(3)}, ...$, at each step we resample each individual variable:

$$\begin{aligned}
\theta_1^{(j)} &\sim p\left(\theta_1 \,|\, \theta_2^{(j-1)}, \theta_3^{(j-1)}, ..., \theta_K^{(j-1)}\right) \\
\theta_2^{(j)} &\sim p\left(\theta_2 \,|\, \theta_1^{(j)}, \theta_3^{(j-1)}, \theta_4^{(j-1)}, ..., \theta_K^{(j-1)}\right) \\
\theta_3^{(j)} &\sim p\left(\theta_3 \,|\, \theta_1^{(j)}, \theta_2^{(j)}, \theta_4^{(j-1)}, ..., \theta_K^{(j-1)}\right) \\
&\vdots \\
\theta_K^{(j)} &\sim p\left(\theta_K \,|\, \theta_1^{(j)}, \theta_2^{(j)}, ..., \theta_{K-1}^{(j)}\right)
\end{aligned}$$

- Because each sampling is one dimensional, simple fast methods can often be used.
- Related to coordinate descent[W] optimisation.

Gibbs Sampling: Function Differencing

- Often times, the conditional probabilities required are cheaply computed via function differencing.
- For simple LDA:

$$p\left(\vec{z},\vec{w}|\vec{\alpha},\vec{\gamma}\right)=\prod_{i=1}^{I}\frac{\text{Beta}_K\left(\vec{\alpha}+\vec{m}_i\right)}{\text{Beta}_K\left(\vec{\alpha}\right)}\prod_{k=1}^{K}\frac{\text{Beta}_V\left(\vec{\gamma}+\vec{n}_k\right)}{\text{Beta}_V\left(\vec{\gamma}\right)}$$

$$\begin{array}{lcll}\vec{m}_i & := & \dim(K) \text{ data counts of topics for doc } i, & \text{s.t. } m_{i,k}=\sum_{l=1}^{L_i}1_{z_{i,l}=k}\\ \vec{n}_k & := & \dim(V) \text{ data counts of words for topic } k, & \text{s.t. } n_{k,w}=\sum_{i=1}^{I}1_{x_{i,l}=w}1_{z_{i,l}=k}\end{array}$$

- Using properties of the Beta/Gamma functions:

$$p\left(z_{i,l}=k|\vec{z}-\{z_{i,l}\},\vec{w},\vec{\alpha},\vec{\gamma}\right)\ \propto\ \left(\alpha_k+m_{i,k}\right)\frac{\gamma_w+n_{k,w}}{\sum_w\gamma_w+n_{k,w}}$$

Gibbs Sampling: Block Sampling

- If two variables are more highly correlated, it makes sense to sample them together.

$$
\begin{aligned}
(\theta_1^{(j)}, \theta_2^{(j)}) &\sim p\left(\theta_1, \theta_2 | \theta_3^{(j-1)}, ..., \theta_K^{(j-1)}\right) \\
\theta_3^{(j)} &\sim p\left(\theta_1 | \theta_1^{(j)}, \theta_2^{(j)}, \theta_4^{(j-1)}, ..., \theta_K^{(j-1)}\right) \\
&\vdots \\
\theta_K^{(j)} &\sim p\left(\theta_1 | \theta_1^{(j)}, \theta_2^{(j)}, ..., \theta_{K-1}^{(j)}\right)
\end{aligned}
$$

- Don't do in general because multi-dimensional sampling is intrinsically harder.
- We usually do it when advantage is also got with function differencing.

Sampling One Real Dimension

- Hyperparameters have real values. When function differencing for them doesn't leave you with a simple sampling form (*e.g.*, Gaussian, Gamma) you need to use a general purpose sampler.
- These are generally much slower than the other cases.

Stochastic gradient descent: If there are not too many of these "hard" dimensions/parameters, this is good enough.

Slice sampling: Easy case when distribution is unimodal. Some optimisations can be tricky. Can fail if posterior highly peaked.

Adaptive Rejection Sampling: Requires distribution be log-concave. Don't try to implement it, use Gilk's C code (1992)

Adaptive Rejection Metropolis Sampling: Extension of Gilk's code. Doesn't require log-concavity.

Sampling One Real Dimension, cont.

Buntine's Collorary to Murphy's Law: if you haven't proven a posterior probability is log-concave (or unimodal), it won't be.

i.e. be careful when sampling.

Non-reversible MCMC

You want to implement split-merge clustering with MCMC. To split a cluster into two parts, a random split is almost certainly going to yield a poor proposal so will be rejected. But a simple heuristic (*e.g.*, one parse greedy clustering) will yield a good split proposal. However, it will be near impossible to develop a corresponding (reverse) merge operation to make the MCMC sampler reversible.

- General MCMC Metropolis-Hastings theory requires samplers be reversible.
- As the example shows, this is extremely difficult for split-merge operations and other complex dimension-jumping moves.
- Jukka Corander's group proved in "Bayesian model learning based on a parallel MCMC strategy," (2006) that MCMC doesn't need to be reversible.
- So simply ignore the reverse operation.

Summary: What You Need to Know

Discrete and conjugate distributions: versions, divisibility, normalising

Dirichlet distribution: basic statistical unit for discrete data

Graphical models: convey the structure of a probability model

Dirichlet-Multinomial: the evidence for a Dirichlet, a distribution itself

LDA topic model: a basic unsupervised component model

Gibbs sampling: various tricks to make it work well and work fast.

Outline

Conjugate Discrete Families

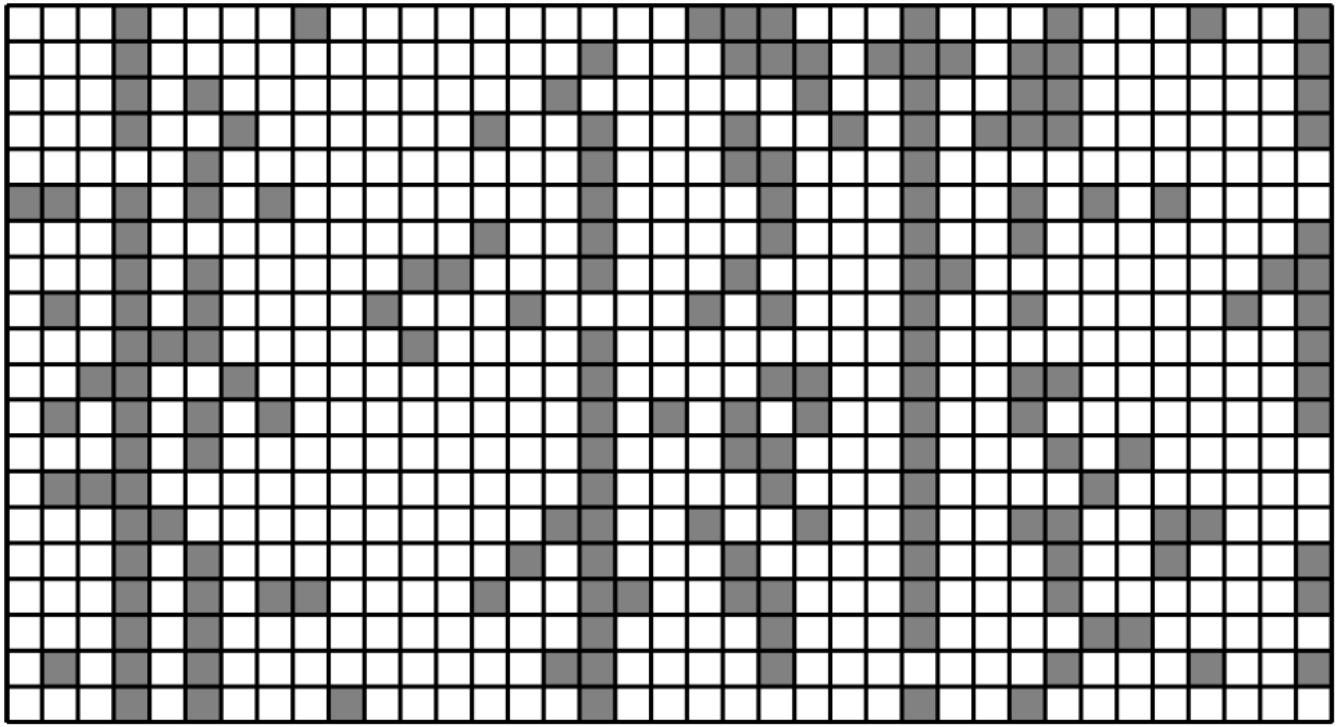
Conjugate Family	$p(x\|\lambda)$	$p(\lambda) \propto$
Bernoulli-Beta	$\lambda^x(1-\lambda)^{1-x}$	$\lambda^{\alpha-1}(1-\lambda)^{\beta-1}\delta_{0<\lambda<1}$
Poisson-Gamma	$\frac{1}{x!}\lambda^x e^{-\lambda}$	$\lambda^{\alpha-1}e^{-\beta\lambda}$
negtve-binomial-Gamma	$\frac{1}{x!}(\lambda)_x\,\rho^x(1-\rho)^\lambda$	$\lambda^{\alpha-1}e^{-\beta\lambda}$

parameters $\alpha, \beta > 0$
$(\lambda)_x$ is rising factorial $\lambda(\lambda+1)...(\lambda+x-1)$

- multinomial-Dirichlet used in LDA
- Poisson-Gamma in some versions of NMF
- Bernoulli-Beta is the basis of IBP
- negative-binomial-Gamma is not quite a conjugate family; the negative-binomial is a "robust" variant of a Poisson

Boolean Matrices

See Griffiths and Ghahramani, 2011.



Each row is a data vector (of features). Each column corresponds to the values (on/off) of one feature. Columns can be infinite but only visualise non-zero features.

Boolean Matrices, cont.

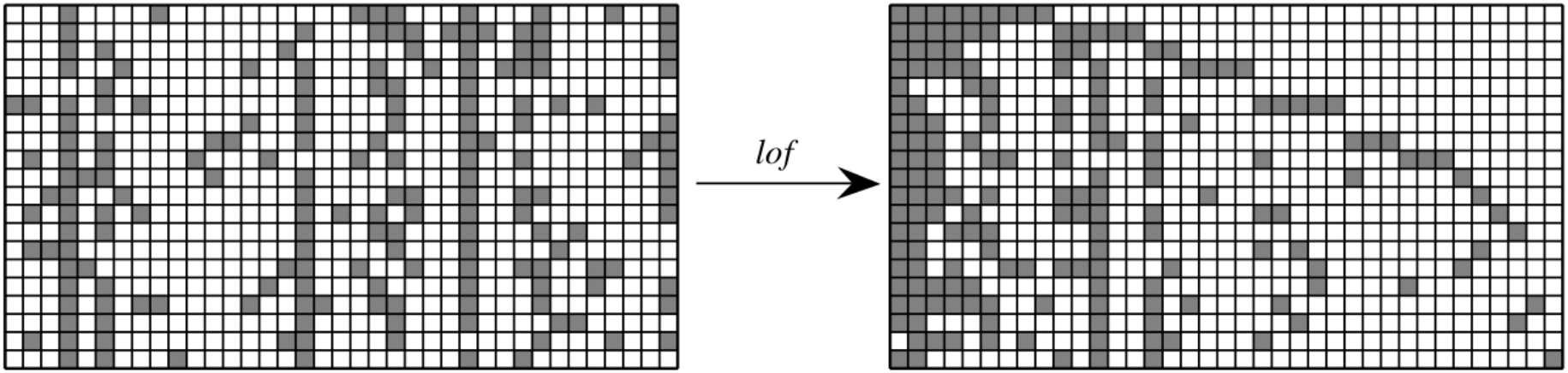


Figure 5: Binary matrices and the left-ordered form. The binary matrix on the left is transformed into the left-ordered binary matrix on the right by the function $lof(\cdot)$. This left-ordered matrix was generated from the exchangeable Indian buffet process with $\alpha = 10$. Empty columns are omitted from both matrices.

- entries in a column are Bernoulli with the same parameter;
- columns are independent;
- Bernoulli probabilities are Beta.

General Discrete Matrices

Boolean matrix
(*e.g.*, Bernoulli-beta)

0	0	1	1	0	1	0
0	1	0	1	0	0	0
1	0	1	0	0	1	1
0	0	1	1	1	0	1
1	0	0	0	1	0	0

count matrix
(*e.g.*, Poisson-gamma)

0	0	1	3	0	2	0
0	4	0	2	0	0	0
1	0	3	0	0	4	2
0	0	4	1	1	0	3
2	0	0	0	1	0	0

Boolean vector matrix
(*e.g.*, Categorical-Dirichlet)

0	0	(1,0,0)	(0,0,1)	0	(0,1,0)	0
0	(1,0,0)	0	(0,1,0)	0	0	0
(1,0,0)	0	(0,0,1)	0	0	(1,0,0)	(0,1,0)
0	0	(1,0,0)	(1,0,0)	(1,0,0)	0	(0,0,1)
(0,1,0)	0	0	0	(1,0,0)	0	0

Infinite Parameter Vectors

- Let $\omega_k \in \Omega$ for $k = 1, ..., \infty$ be index points for an infinite vector.
- Each ω_k indexes a column.
- Have infinite parameter vector $\vec{\lambda}$ represented as
  $\lambda(\omega) = \sum_{k=1}^{\infty} \lambda_k \delta_{\omega_k}(\omega)$, for $\lambda_k \in (0, \infty)$.
- Acts as a function on ω (assuming all ω_k are distinct):

$$\vec{\lambda}(\omega) = \begin{cases} \lambda_k & \omega \equiv \omega_k \\ 0 & \text{otherwise} \end{cases}$$

See 2014 ArXiv paper by Lancelot James
(`http://arxiv.org/abs/1411.2936`).

Infinite Discrete Vectors

Modelling this in the general case (discrete data rather than just Booleans):

- Have I discrete feature vectors $\vec{x}_i$ represented as $x_i(\omega) = \sum_{k=1}^{\infty} x_{i,k} \delta_{\omega_k}(\omega)$.
- Generate pointwise from $\vec{\lambda}$ using discrete distribution $p(x_{i,k}|\lambda_k)$.
- Corresponds to a row.
- Each row should only have a finite number of non-zero entries.
- So expect $\sum_{k=1}^{\infty} 1_{x_{i,k} \neq 0} < \infty$ for each i.
- This means we need $\sum_{k=1}^{\infty} \lambda_k < \infty$:
  - assuming lower λ_k makes $x_{i,k}$ more likely to be zero,
  - e.g., for the Bernoulli, $\mathbb{E}[x_{i,k}] = \lambda_k$

ASIDE: Infinite Vectors: Arbitrarily Reordering

- Remember the "left ordered form".
- Can arbitrarily reorder dimensions k since all objects have term $\sum_{k=1}^{\infty}(\cdot)$.
- So for any reordering σ,

$$\sum_{k=1}^{\infty} \lambda_k \delta_{\omega_k}(\theta) \;=\; \sum_{k=1}^{\infty} \lambda_{\sigma(k)} \delta_{\omega_{\sigma(k)}}(\theta) \;.$$

- Don't know which dimensions k non-zero so reorder afterwards so only non-zero dimensions are included.

ASIDE: Improper Uniform Prior for Gaussian Data

- Suppose we want a "uniform" prior on location μ on the real line. A constant prior on μ must be over a finite domain:

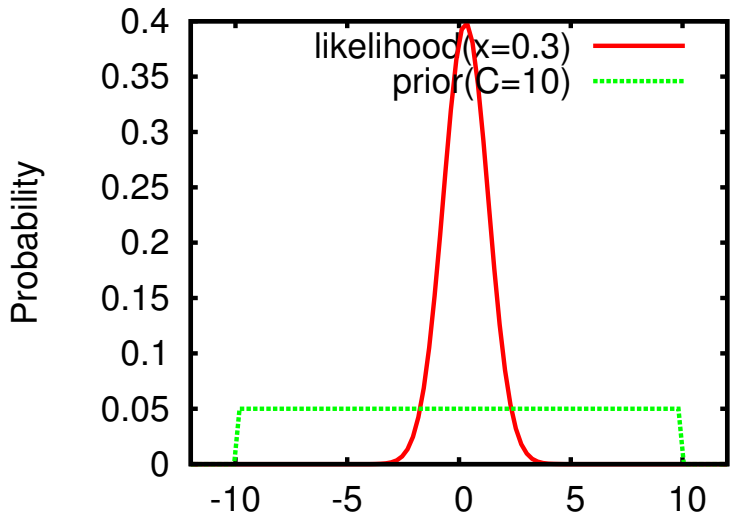
$$p(\mu) \sim \frac{1}{2C} \qquad \text{for } \mu \in [-C, C]$$

- Consider data x_1 with Gaussian likelihood $p(x|\mu) = \frac{1}{\sqrt{2\pi}} e^{(x-\mu)^2}$.
  - for C large enough, this is possible

As $C \to \infty$, the corresponding posterior is

- $$p(\mu|x_1) = \frac{1}{\sqrt{2\pi}} e^{(x_1-\mu)^2} .$$

but the prior is not proper.



ASIDE: Improper Prior

Informally, an improper prior is not a proper distribution. It is a measure that cannot be normalised. It is constructed:

- as the limit of a sequence of proper distributions,
- where the corresponding limit of posteriors from any one data point is a proper distribution.

Generating Infinite Vectors

- Formally modelled using Poisson processes:

  Have Poisson process with points (λ, ω) *on domain* $(0, \infty) \times \Omega$ *with rate* $p(\lambda|\omega)d\lambda G(d\omega)$.

- For infinite length vectors want infinite number of points ω_k sampled by the Poisson process, so rate doesn't normalise:

$$\int_0^\infty \int_\Omega p(\lambda|\omega)\mathrm{d}\lambda G(\mathrm{d}\omega) = \infty$$

- To expect $\sum_{k=1}^\infty \lambda_k < \infty$, want

$$\mathbb{E}\left[\sum_{k=1}^\infty \lambda_k\right] = \int_0^\infty \int_\Omega \lambda p(\lambda|\omega)\mathrm{d}\lambda G(\mathrm{d}\omega) < \infty$$

- Parameters to the "distribution" (Poisson process rate) for λ would control the expected number of non-zero $x_{i,k}$.

Bernoulli-Beta Process (Indian Buffet Process)

- infinite Boolean vectors $\vec{x}_i$ with a finite number of 1's;
- each parameter λ_k is an independent probability,

$$p(x_{i,k}|\lambda_k) \;=\; \lambda_k^{x_{i,k}}(1-\lambda_k)^{1-x_{i,k}}$$

- to have finite 1's, require $\sum_k \lambda_k < \infty$
- improper prior (Poisson process rate) is the 3-parameter Beta process

$$p(\lambda|\alpha,\beta,\theta) \;=\; \theta\lambda^{-\alpha-1}(1-\lambda)^{\alpha+\beta-1}$$

(some versions add additional constants with θ)

- is in improper Beta because seeing "1" makes it proper:

$$\int_{\lambda=0}^{1} p(x=1|\lambda)p(\lambda)\mathrm{d}\lambda \;=\; \theta\,\mathrm{Beta}(1-\alpha,\alpha+\beta)$$

Outline

Conjugate Discrete Processes

Each conjugate family has a corresponding non-parametric version:

- Uses the improper versions of the prior $p(\lambda|\omega)$
  e.g. for Gamma, Beta, Dirichlet
- Want to generate a countably infinite number of λ but have almost all infinitesimally small.
- Theory done with Poisson processes, see 2014 ArXiv paper by Lancelot James (`http://arxiv.org/abs/1411.2936`).
- Presention here uses the more informal language of "improper priors," but the correct theory is Poisson processes.

Conjugate Discrete Processes, cont.

Non-parametric versions of models for discrete feature vectors:

Process Name	$p(x\|\lambda)$	$p(\lambda)$
Poisson-Gamma	$\frac{1}{x!}\lambda^x e^{-\lambda}$	$\theta\lambda^{-\alpha-1}e^{-\beta\lambda}$
Bernoulli-Beta	$\lambda^x(1-\lambda)^{1-x}$	$\theta\lambda^{-\alpha-1}(1-\lambda)^{\alpha+\beta-1}\delta_{0<\lambda<1}$
negtve-binomial-Gamma	$\frac{1}{x!}(\lambda)_x\rho^x(1-\rho)^\lambda$	$\theta\lambda^{-\alpha-1}e^{-\beta\lambda}$

$\beta, \theta > 0$
$0 \leq \alpha < 1$

- In common they make the power of λ lie in $(-2, -1]$ to achieve the "improper prior" effect.
- Term θ is just a general proportion to uniformally increase number of λ_k's in any region.
- Whereas α and β control the relative size of the λ_k's.

Deriving the Posterior Marginal

- Have Poisson process with points (λ, ω) on domain $(0, \infty) \times \Omega$ with rate $p(\lambda|\omega)\mathrm{d}\lambda G(\mathrm{d}\omega)$.
- Given λ, probability of I samples with at least one non-zero entry is

$$\left(1 - p(x_{i,k} = 0|\lambda)^I\right) \ .$$

- By Poisson process theory, expectation of this is rate of generating a feature k with non-zero in I data:

$$\Psi_I = \int_\Omega \int_0^\infty \left(1 - p(x_{i,k} = 0|\lambda)^I\right) \rho(\lambda|\omega)\mathrm{d}\lambda G_0(\mathrm{d}\omega_k)$$

- Call Ψ_I the Poisson non-zero rate, a function of I.
- With I vectors, number of non-zero dimensions K is Poisson with rate Ψ_I, having probability

$$\frac{1}{K!} e^{-\Psi_I} \Psi_I^K \ .$$

Deriving the Posterior Marginal, cont.

- With I vectors, number of non-zero dimensions K is Poisson with rate Ψ_I, having probability
$$\frac{1}{K!} e^{-\Psi_I} \Psi_I^K \ .$$
- Take particular dimension ordering (remove $\frac{1}{K!}$) and replace "not all zero" by actual data, $x_{i,1}, ..., x_{i,K}$ to get:
$$e^{-\Psi_I} \prod_{k=1}^{K} p(x_{1,k}, ..., x_{i,k}, \omega_w) \ .$$
- Expand using model to get posterior marginal:
$$p(\vec{x}_1, ..., \vec{x}_I, \vec{\omega}) = e^{-\Psi_I} \prod_{k=1}^{K} \left(\int_0^\infty \left(\prod_{i=1}^{I} p(x_{i,k}|\lambda) \right) \rho(\lambda|\omega) \mathrm{d}\lambda \right) G_0(\mathrm{d}\omega_k)$$

Outline

Bernoulli-Beta Process (Indian Buffet Process)

- The Poisson non-zero rate
  trick: use $1 - y^I = (1 - y) \sum_{i=0}^{I} y^i$

$$\Psi_I \;=\; \theta\Gamma(1-\alpha)\sum_{i=0}^{I}\frac{\Gamma(\beta+\alpha+i)}{\Gamma(\beta+1+i)}\;.$$

- The marginal for the k-th dimension

$$\int_0^\infty \left(\prod_{i=1}^{I} p(x_{i,k}|\lambda)\right) \rho(\lambda|\omega)\mathrm{d}\lambda \;=\; \theta\,\mathrm{Beta}(c_k - \alpha, I - c_k + \alpha + \beta)$$

  where c_k is times dimension k is "on," so $c_k = \sum_{i=1}^{I} x_{i,k}$.

Bernoulli-Beta Process, cont.

Marginal posterior:

$$\theta^K \, e^{-\theta\Gamma(1-\alpha)\sum_{i=0}^{I}\frac{\Gamma(\beta+\alpha+i)}{\Gamma(\beta+1+i)}} \prod_{k=1}^{K} \text{Beta}(c_k - \alpha, I - c_k + \alpha + \beta)$$

where c_k is times dimension k is "on," so $c_k = \sum_{i=1}^{I} x_{i,k}$.

Gibbs sampling $x_{i,k}$: affect of this term at least, is thus simple.

Sampling hyperparameters: posterior of θ is Gamma; posterior for β is log-concave so sampling "easier".

Poisson-Gamma Process

- The Poisson non-zero rate
  trick: use the Laplace exponent from Poisson process theory

$$\Psi_I \;=\; \theta \frac{\Gamma(1-\alpha)}{\alpha}\left((I+\beta)^{\alpha}-\beta^{\alpha}\right) \;.$$

- The marginal for the k-th dimension

$$\int_0^{\infty}\left(\prod_{i=1}^{I} p(x_{i,k}|\lambda)\right)\rho(\lambda|\omega)\mathrm{d}\lambda \;=\; \theta\left(\prod_{i=1}^{I}\frac{1}{x_{i,k}!}\right)\frac{\Gamma(x_{\cdot,k}-\alpha)}{(I+\beta)^{x_{\cdot,k}-\alpha}}$$

  where $x_{\cdot,k}=\sum_{i=1}^{I} x_{i,k}$.

Poisson-Gamma Process, cont

Marginal posterior:

$$\theta^K e^{-\theta \frac{\Gamma(1-\alpha)}{\alpha}((I+\beta)^\alpha - \beta^\alpha)} \left(\prod_{i=1,k=1}^{I,K} \frac{1}{x_{i,k}!} \right) \prod_{k=1}^{K} \frac{\Gamma(x_{\cdot,k} - \alpha)}{(I+\beta)^{x_{\cdot,k}-\alpha}}$$

where $x_{\cdot,k} = \sum_{i=1}^{I} x_{i,k}$.

Gibbs sampling the $x_{i,k}$: affect of this term at least, is thus simple.

Sampling hyperparameters: posterior of θ is Gamma; posterior of β is unimodal (and no other turning points) with simple closed form for MAP.

Negative-Binomial-Gamma Process

- Series of papers for this case by Mingyuan Zhou and colleages.
- The Poisson non-zero rate
  trick: use the Laplace exponent from Poisson process theory

$$\Psi_l = \theta \frac{\Gamma(1-\alpha)}{\alpha} \left(\left(l \log \left(\frac{1}{1-p} \right) + \beta \right)^{\alpha} - \beta^{\alpha} \right) .$$

- The marginal for the k-th dimension

$$\int_0^{\infty} \left(\prod_{i=1}^{I} p(x_{i,k}|\lambda, p) \right) \rho(\lambda|\omega) \mathrm{d}\lambda$$

$$= p^{x_{\cdot,k}} \left(\prod_{i=1}^{I} \frac{1}{x_{i,k}!} \right) \int_0^{\infty} (1-p)^{I\lambda} \left(\prod_{i=1}^{I} (\lambda)_{x_{i,k}} \right) \rho(\lambda) \mathrm{d}\lambda$$

- Gibbs sampling the $x_{i,k}$ is more challenging.
  - keep λ as a latent variable (posterior is log concave);
  - use approximation $(\lambda)_x \approx \lambda^{t^*} S^x_{t^*,0}$ where $t^* = \mathrm{argmax}_{t \in [1,x)} \lambda^t S^x_{t,0}$.

ASIDE: Simple, Fast Hierarchical IBP

James' more general theory allows more creativity in construction.

Bernoulli-Beta-Beta process

- model is a hierarchy of Bernoulli-Beta processes
- infinite feature vector $\vec{\lambda}$ is a Beta Process as before;
- these varied with point-wise Beta distributions to create a set of parent nodes $\vec{\psi}_j$, so $\psi_{j,k} \sim \text{Beta}(\alpha\lambda_{j,k}, \alpha(1-\lambda_{j,k}))$
- discrete features ordered in a hierarchy below nodes j so $x_{i,k} \sim \text{Bernoulli}(\psi_{j,k})$ for j the parent of node i.
- Use hierarchical Dirichlet process techniques to implement efficiently.

Summary: What You Need to Know

Infinite Feature Vectors: mathematical notation for representing unbounded vectors

Generalised IBP: templates based on standard discrete conjugate families

Marginal Posteriors for IBP: once you have formulas, allows easy Gibbs sampling

Hierarchical IBP: easily set up using PYP theory.

Versions of the generalised IBP are at the leading edge of new methods in topic models.

Outline

Definition: Species Sampling Model

Definition of a species sampling model

Have a probability vector $\vec{p}$ (so $\sum_{k=1}^{\infty} p_k = 1$), and a domain Θ and a countably infinite sequence of elements $\{\theta_1, \theta_2, ...\}$ from Θ.
A species sampling model (SSM) draws a sample θ according to the distribution

$$p(\theta) = \sum_{k=1}^{\infty} p_k \delta_{\theta_k}(\theta) .$$

- sample θ_k with probability p_k
- if $\forall_k \theta \neq \theta_k$, then $p(\theta) = \sum_k p_k 0 = 0$
- if $\forall_{k\,:\,k\neq l} \theta_l \neq \theta_k$, then $p(\theta_l) = \sum_k p_k \delta_{k=l} = p_l$

Species Sampling Model, cont.

SSM defined as:

$$p(\theta) \;=\; \sum_{k=1}^{\infty} p_k \delta_{\theta_k}(\theta) \;.$$

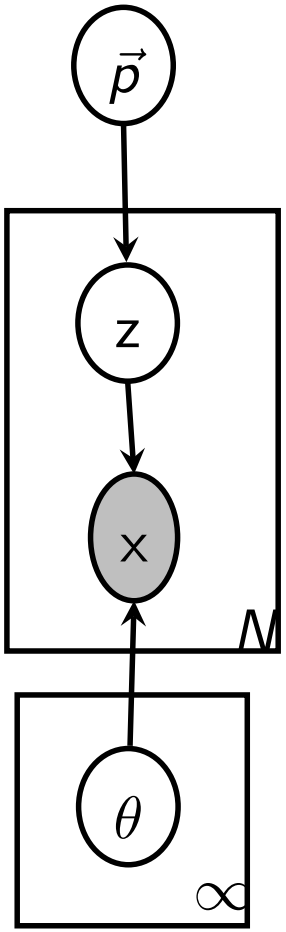
- the indices themselves in $(\sum_{k=1}^{\infty} \cdot)$ are irrelevant, so for any reordering σ,

$$p(\theta) \;=\; \sum_{k=1}^{\infty} p_{\sigma(k)} \delta_{\theta_{\sigma(k)}}(\theta) \;;$$

- to create an SSM, one needs a sequence of values θ_k
  - usually we generate these independently according to some base distribution (usually $H(\cdot)$) so $\theta_k \sim H(\cdot)$
- to create an SSM, one also needs a vector $\vec{p}$;
  - this construction is where all the work is!

Using an SSM for a Mixture Model

Classic MM



On the left

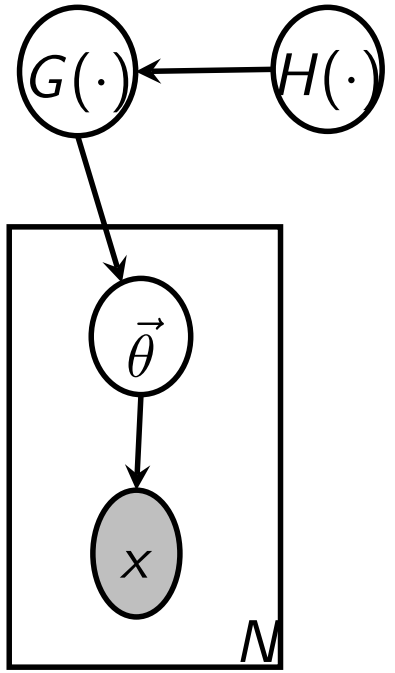
$$
\begin{aligned}
\vec{p} &\sim \text{SSM-p}(\cdot) \\
\vec{\theta}_k &\sim H(\cdot) && \forall_{k=1,\ldots,K} \\
z_n &\sim \vec{p} && \forall_{n=1,\ldots,N} \\
x_n &\sim f\left(\vec{\theta}_{z_n}\right) && \forall_{n=1,\ldots,N}
\end{aligned}
$$

Versus, on the right

$$
\begin{aligned}
G(\cdot) &\sim \text{SSM}\left(H(\cdot)\right) \\
\vec{\theta}_n &\sim G(\cdot) && \forall_{k=1,\ldots,N} \\
x_n &\sim f\left(\vec{\theta}_n\right) && \forall_{n=1,\ldots,N}
\end{aligned}
$$

where $G(\vec{\theta})$ is an SSM, including a vector $\vec{p}$.

SSM MM



Infinite Probability Vectors $\vec{p}$

- The number of $p_k > \delta$ must be less than $(1/\delta)$.

  e.g. there can be no more than 1000 p_k greater than 0.001.
- The value of p_{58153} is almost surely infinitessimal.
  - and for $p_{9356483202}$, *etc.*
- But some of the p_k must be larger and significant.
- It is meaningless to consider a p_k without:
  - defining some kind of ordering on indices,
  - only considering those greater than some δ, **or**
  - ignoring the indices and only considering the partitions of data induced by the indices.

ASIDE: Schemes for Generating $\vec{p}$

There are general schemes (but also more) for sampling infinite probability vectors:

Normalised Random Measures: sample an independent set of weights w_k (a "random measure") using for instance, a Poisson process, and then normalise, $p_k = \frac{w_k}{\sum_{k=1}^{\infty} w_k}$.

Predictive Probability Functions: generalises the famous "Chinese Restaurant Process" we will cover later. See Lee, Quintana, Müller and Trippa (2013).

Stick-Breaking Construction: commonly used definition for the Pitman-Yor process we will consider later. See Ishwaran and James (2001).

Outline

Partitions

Definition of partition

A partition of a set$^{\mathcal{W}}$ P of a countable set X is a mutually exclusive and exhaustive set of non-empty subsets of X. The partition size of P is given by the number of sets $|P|$.

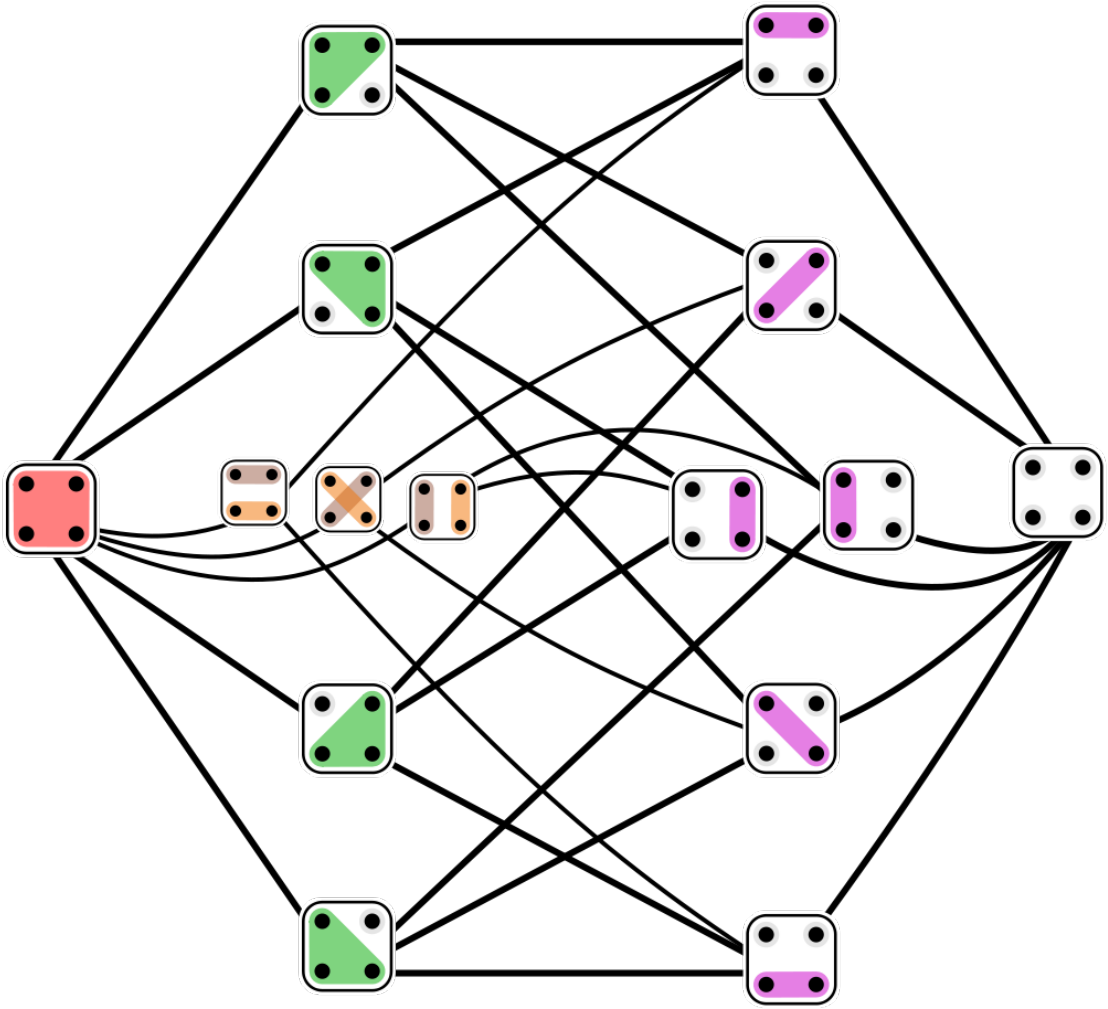
Consider partitions of the set of letters $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o\}$:

Candidate	Legality
{a,d,k,n},{b,f,h,i,j},{c,e,l,m,o},{g}	OK
{a,d,k,n},{b,f,h,i,j},{c,e,l,m,o},{g,k}	no, 'k' duped
{a,d,k,n},{b,f,h,i,j},{c,e,l,m,o}	no, not exhaustive
{a,d,k,n},{b,f,h,i,j},{c,e,l,m,o},{g},{}	no, an empty set

Partitions over $\{a, b, c\}$

partition P	$\{\{a, b, c\}\}$	$\{\{a, b\}, \{c\}\}$	$\{\{a, c\}, \{b\}\}$	$\{\{a\}, \{b, c\}\}$	$\{\{a\}, \{b\}, \{c\}\}$
indices	$(1, 1, 1)$	$(1, 1, 2)$	$(1, 2, 1)$	$(1, 2, 2)$	$(1, 2, 3)$
size $\|P\|$	1	2	2	2	3
counts $\vec{n}$	(3)	$(2, 1)$	$(2, 1)$	$(1, 2)$	$(1, 1, 1)$

ASIDE: All partitions of 4 objects



Note: space of partitions forms a lattice.

A Sample of a SSM Induces a Partition

Suppose we have a sample of size $N = 12$ taken from an infinite mixture (for simplicity, we'll label data as 'a', 'b', ...):

a,c,a,d,c,d,a,b,g,g,a,b

This can be represented as follows:

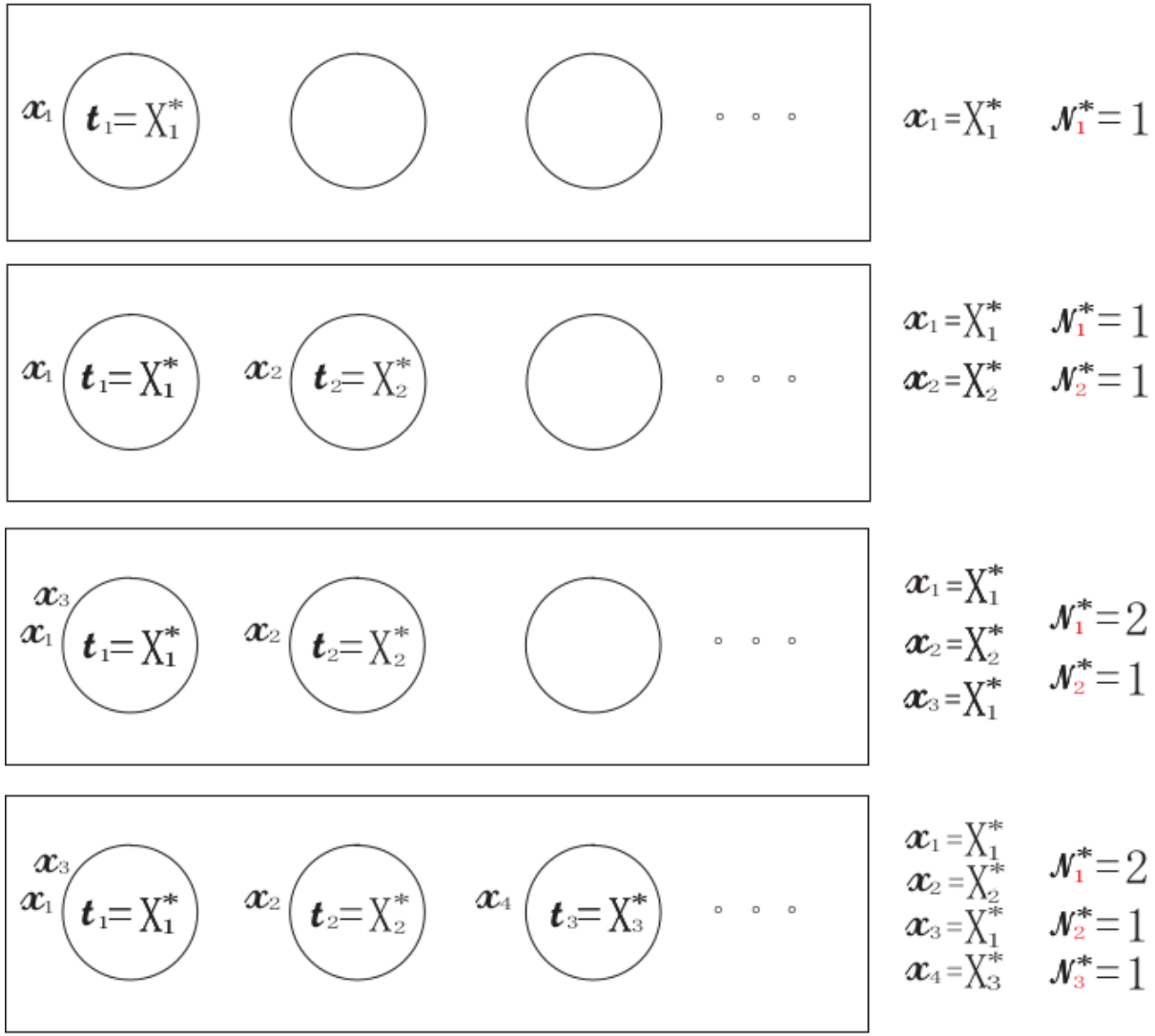
1,2,1,3,2,3,1,4,5,5,1,4

where index mappings are: a=1, c=2, d=3, b=4, g=5.

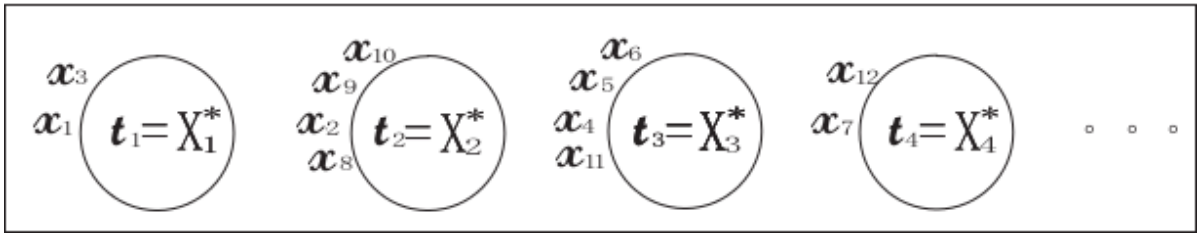
- The sample induces a partition of N objects.
- Index mappings can be arbitrary, but by convention we index data as it is first seen as 1,2,3,...
- This convention gives the size-biased ordering for the partition,
  - because the first data item seen is more likely to have the largest p_k, the second data item seen is more likely to have the second largest p_k, *etc.*

Outline

ASIDE: "Chinese Restaurant" Sampling



ASIDE: CRP Terminology



Restaurant: single instance of a CRP, roughly like a Dirichlet-multinomial distribution.

Customer: one data point.

Table: cluster of data points sharing the one sample from $H(.)$.

Dish: the data value corresponding to a particular table; all customers at the table have this "dish".

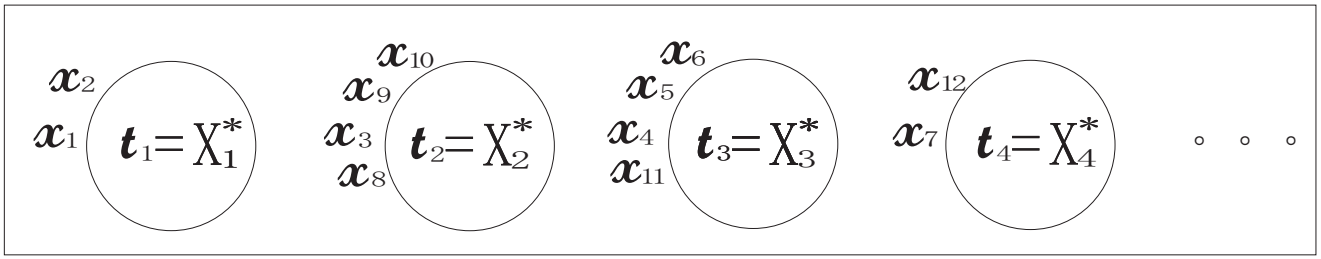
Table count: number of customers at the table.

Seating plan: full configuration of tables, dishes and customers.

ASIDE: CRP Example

CRP with base distribution $H(\cdot)$:

$$p(x_{N+1} \mid x_{1:N}, d, \alpha, H(\cdot)) = \frac{\alpha + Kd}{N+\alpha} H(x_{N+1}) + \sum_{k=1}^{K} \frac{n_k - d}{N+\alpha} \delta_{X_k^*}(x_{N+1}) ,$$



$$p(x_{13} = X_1^* \mid x_{1:12}, \ldots) = \frac{2-d}{12+\alpha} \qquad p(x_{13} = X_2^* \mid x_{1:12}, \ldots) = \frac{4-d}{12+\alpha}$$

$$p(x_{13} = X_3^* \mid x_{1:12}, \ldots) = \frac{4-d}{12+\alpha} \qquad p(x_{13} = X_4^* \mid x_{1:12}, \ldots) = \frac{2-d}{12+\alpha}$$

$$p(x_{13} = X_5^* \mid x_{1:12}, \ldots) = \frac{\alpha + 4d}{12+\alpha} H(X_5^*)$$

Evidence for the CRP

It does show how to compute evidence only when $H(\cdot)$ is non-discrete.

$$
\begin{aligned}
& p(x_1, ..., x_N | N, CRP, d, \alpha, H(\cdot)) \\
&= \frac{\alpha(d+\alpha)...((K-1)d+\alpha)}{\alpha(1+\alpha)...(N-1+\alpha)} \prod_{k=1}^{K} ((1-d)...(n_k-1-d)H(X_k^*)) \\
&= \frac{(\alpha|d)_K}{(\alpha)_N} \prod_{k=1}^{K} (1-d)_{n_k-1} H(X_k^*) \ ,
\end{aligned}
$$

where there are K distinct data values X_1^*, ...X_K^*.

For the DP version, $d = 0$, this simplifies

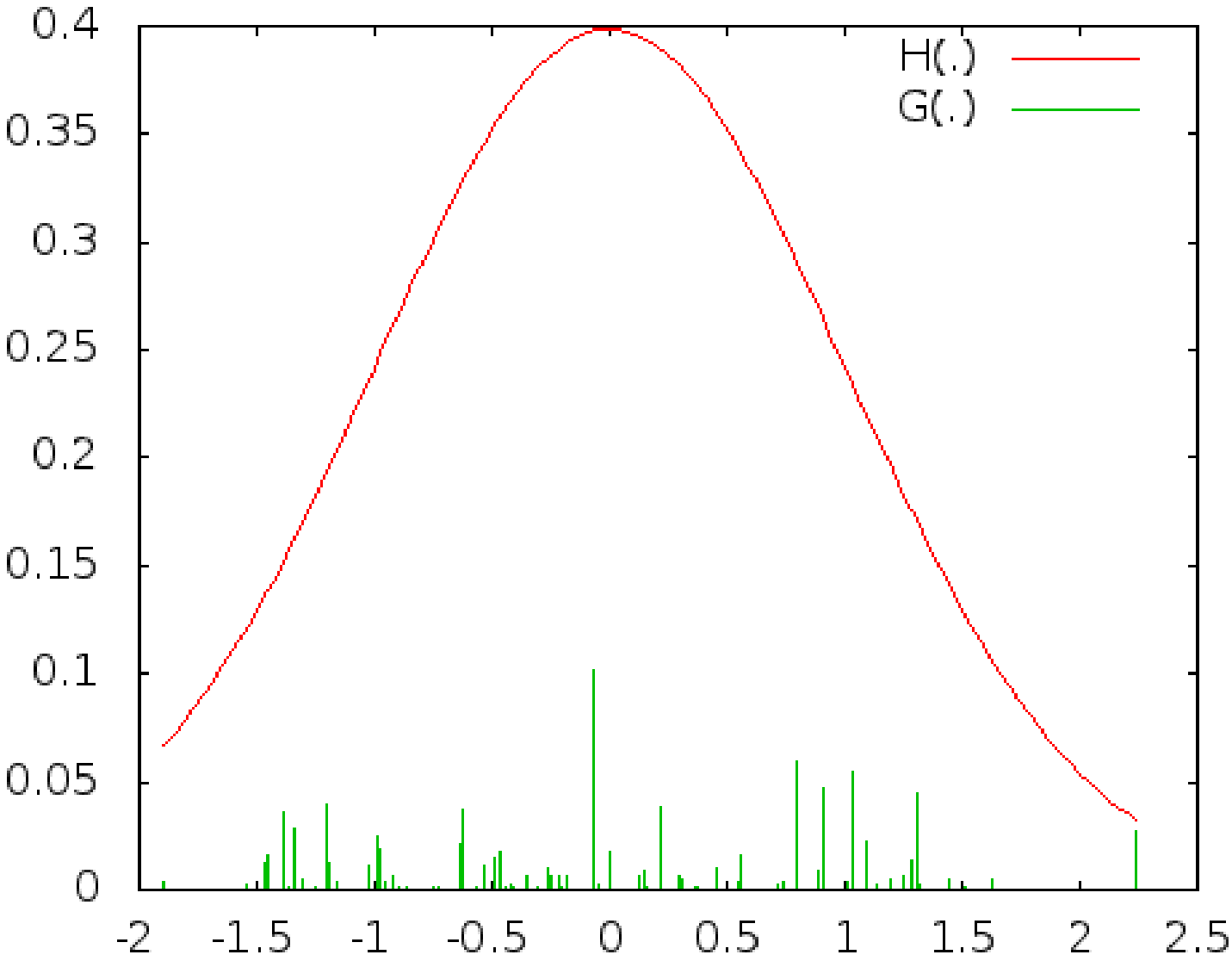
$$
\frac{\alpha^K}{(\alpha)_N} \prod_{k=1}^{K} \Gamma(n_k) H(X_k^*) \ .
$$

Outline

PYP and DP

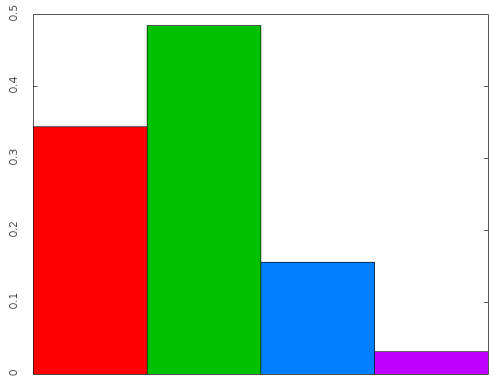
- The Pitman-Yor Process (PYP) has three arguments PYP $(d, \alpha, H(\cdot))$ and the Dirichlet Process (DP) has two arguments DP $(\alpha, H(\cdot))$:
  - Discount d is the Zipfian slope for the PYP.
  - Concentration α is inversely proportional to variance.
  - Base distribution $H(\cdot)$ that seeds the distribution and is the mean.
  - *e.g.*, as $\alpha \to \infty$, a sample from them gets closer to $H(\cdot)$.
- They return an SSM, $p(\theta) = \sum_{k=1}^{\infty} p_k \delta_{\theta_k}(\theta)$, where θ_k are independently and identically distributed according to the base distribution $H(\cdot)$.
- They return a distribution on the same space as the base distribution (hence are a functional).
  - fundamentally different depending on whether $H(\cdot)$ is discrete or not.
- PYP originally called "two-parameter Poisson-Dirichlet process" (Ishwaran and James, 2003).

Example: $G(\cdot) \sim \mathrm{DP}(1, \mathrm{Gaussian}(0, 1))$

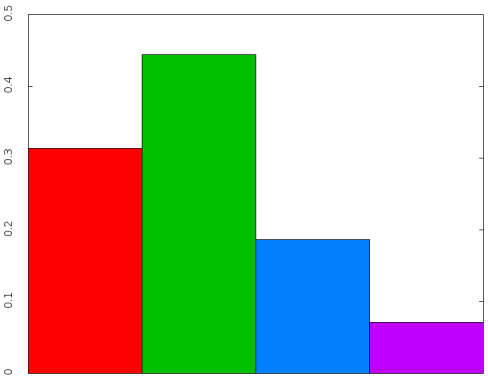


Example: DP on a 4-D vector

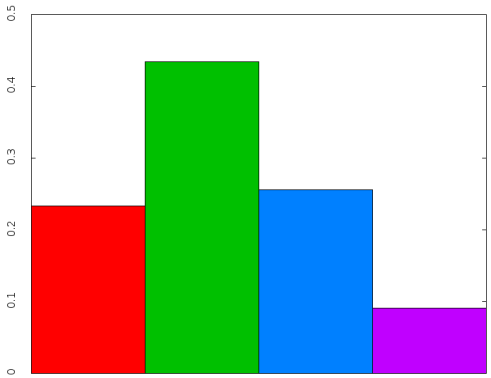
$\vec{p}_0$



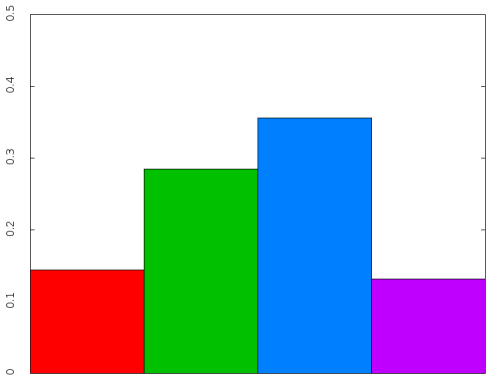
$\vec{p}_1 \sim \text{DP}(500, \vec{p}_0)$



$\vec{p}_2 \sim \text{DP}(5, \vec{p}_0)$



$\vec{p}_3 \sim \text{DP}(0.5, \vec{p}_0)$



Definitions

There are several ways of defining a *Pitman-Yor Process* of the form PYP$(d, \alpha, H(\cdot))$.

- Generate a $\vec{p}$ with a GEM(d, α), then form an SSM by independently sampling $\theta_k \sim H(\cdot)$ (Pitman and Yor, 1997).
- Generate a $\vec{p}$ with an ImproperDirichlet(d, α), then form an SSM by independently sampling $\theta_k \sim H(\cdot)$.
- Proport its existence by saying it has posterior sampler given by a CRP$(d, \alpha, H(\cdot))$.

There is another way of defining a *Dirichlet Process* of the form DP$(\alpha, H(\cdot))$.

- As a natural extension to the Dirichlet in non-discrete or countably infinite domains (see "formal definition" in Wikipedia).

Dirichlet Process

- When applied to a finite probability vector $\vec{\mu}$ of dimension K, the DP and the Dirichlet are identical:

$$\text{Dirichlet}_K(\alpha, \vec{\mu}) \;=\; \text{DP}(\alpha, \vec{\mu}) \;.$$

- Thus in many applications, the use of a DP is equivalent to the use of a Dirichlet.
- Why use the DP then?
  - Hierarchical Dirichlets have fixed point MAP solutions, but more sophisticated reasoning is not always possible.
  - Hierarchical DPs have fairly fast samplers (as we shall see).
  - MAP solutions for hierarchical Dirichlets could be a good way to "burn in" samplers.

Outline

How Many Species/Tables are There?

- Consider just the case where the sample $\vec{x}$ of size N has just K species (or tables for the CRP), denoted as $|\vec{x}| = K$.
- What is the expected distribution on K?
- Easily sampled using the CRP.
- Can also be found in closed form.

PYP Species Count Posterior

Consider just the case where the sample $\vec{x}$ of size N has just K species, denoted as $|\vec{x}| = K$.

$$p(|\vec{x}| = K|d, \alpha, N, \text{PYP}) \quad \propto \quad \frac{(\alpha|d)_K}{(\alpha)_N} \sum_{\vec{x}\,:\,|\vec{x}|=K} \prod_{k=1}^{K} (1-d)_{n_k-1}$$

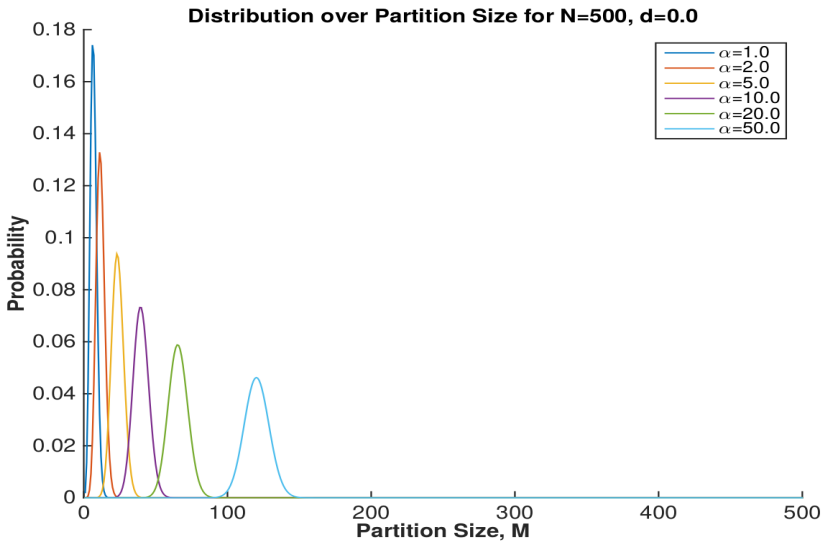
$$\text{Define} \qquad \mathcal{S}^N_{K,d} \quad := \quad \sum_{\vec{x}\,:\,|\vec{x}|=K} \prod_{k=1}^{K} (1-d)_{n_k-1} \ ,$$

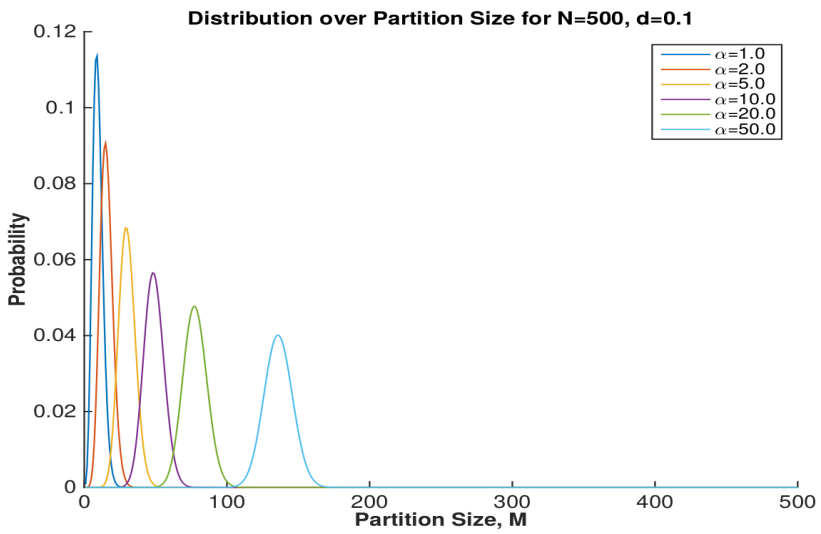
$$\text{then} \quad p(|\vec{x}| = K|d, \alpha, N, \text{PYP}) \quad = \quad \frac{(\alpha|d)_K}{(\alpha)_N} \mathcal{S}^N_{K,d} \ .$$

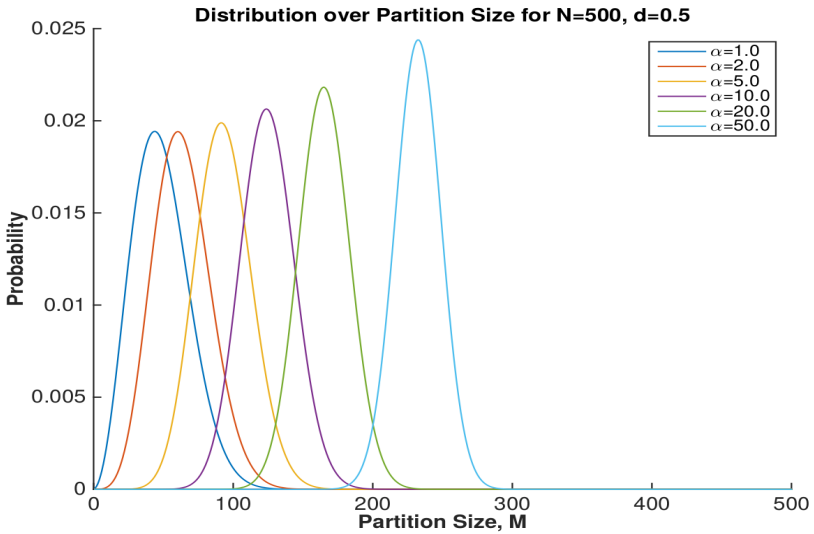
The $\mathcal{S}^N_{K,d}$ is a generalised Stirling number of the second kind, with many nice properties. Is easily tabulated so $O(1)$ to compute.

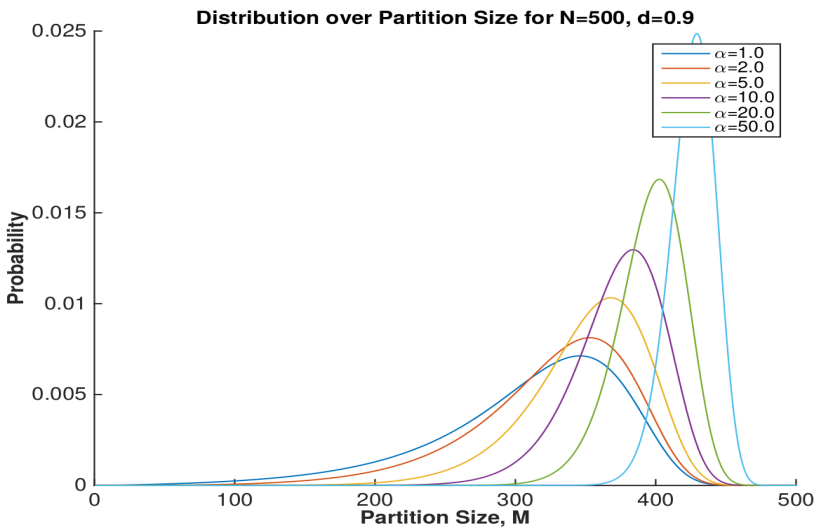
See Buntine & Hutter 2012, and for code the MLOSS project `libstb`.

How Many Species/Tables are There When $N = 500$?







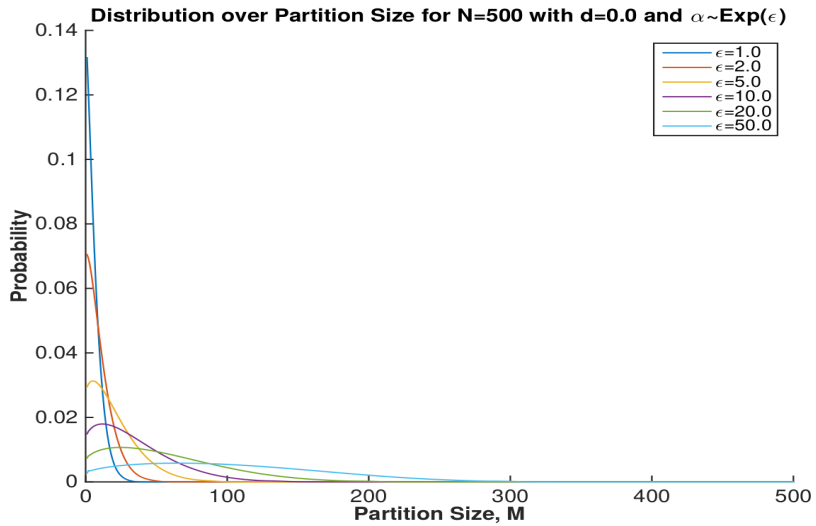


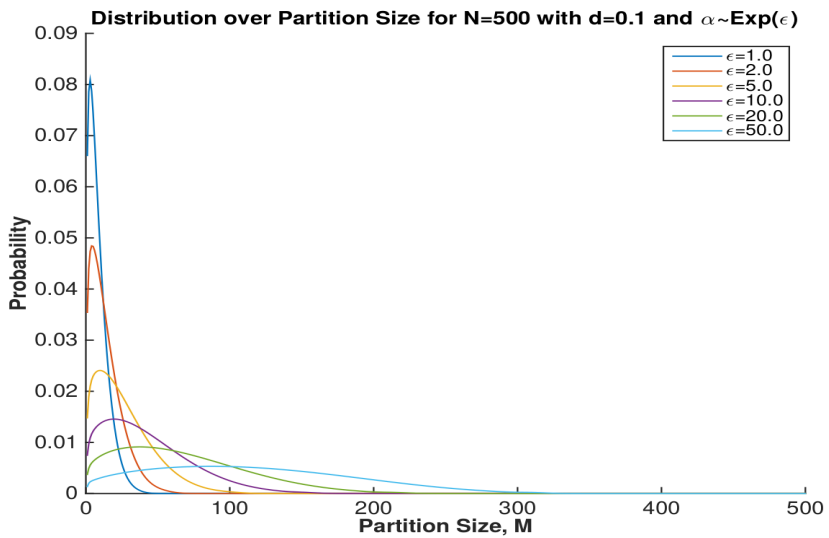
Posterior probability on K given $N = 500$ and different d, α.

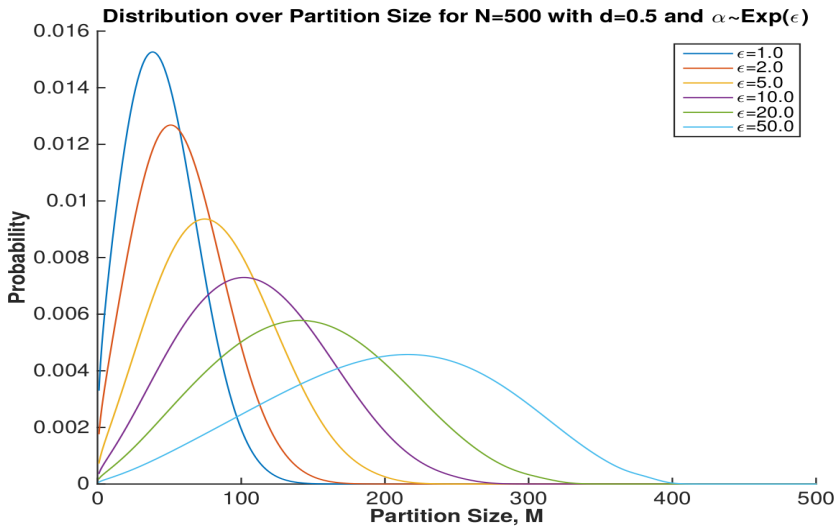
Number of Species/Tables

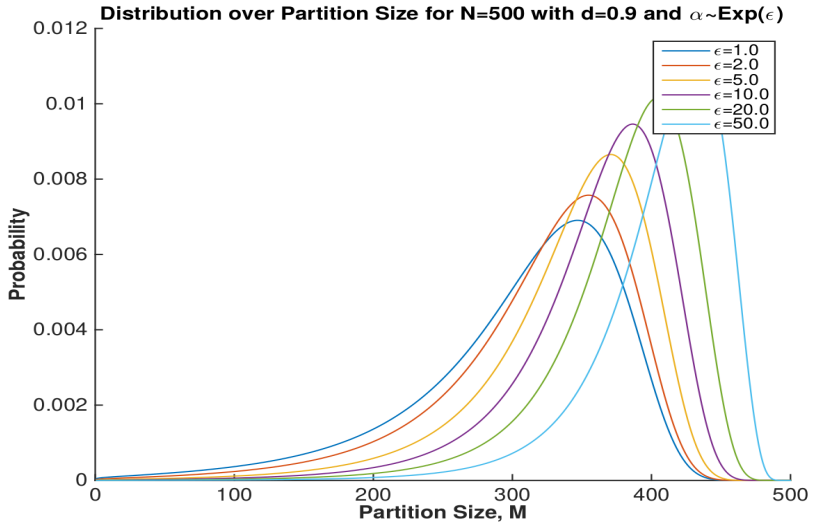
- The number of species/tables varies dramatically depending on the discount and the concentration.
- Is approximately Gaussian with a smallish standard-deviation.
- In applications, we should probably sample the posterior for discount and/or the concentration.
- Note concentration has fast effective posterior samplers, but sampling discount is slow when using Stirling numbers.

How Many Species/Tables are There, Sampling α?









As before with different d but now sampling $\alpha \sim \text{Exp}(\epsilon)$.

Sampling Concentration and Discount

- In some cases the concentration and/or discount can be well chosen apriori:
  - the Stochastic Memoizer (Wood *et al.*, ICML 2009) uses a particular set for a hierarchy on text,
  - text is known to work well with discount $d \approx 0.5 - 0.7$,
  - topic proportions in LDA known to work well with discount $d = 0.0$.
- The concentration samples very nicely using a number of schemes.
  - slice sampling or adaptive rejection sampling,
  - auxiliary variable sampling (Teh *et al.*, 2006) when $d = 0$,

  $\longrightarrow$ usually improves performance, so do by default.
- The discount is expensive to sample (the generalised Stirling number tables introduced later need to be recomputed), but can be done with slice sampling or adaptive rejection sampling.

Summary: What You Need to Know

Species Sampling Model: SSM returns a discrete number of points from a domain

Partition: mixture models partition data, indexes are irrelevant

Chinese Restaurant Process: CRP is the marginalised posterior sampler for PYP

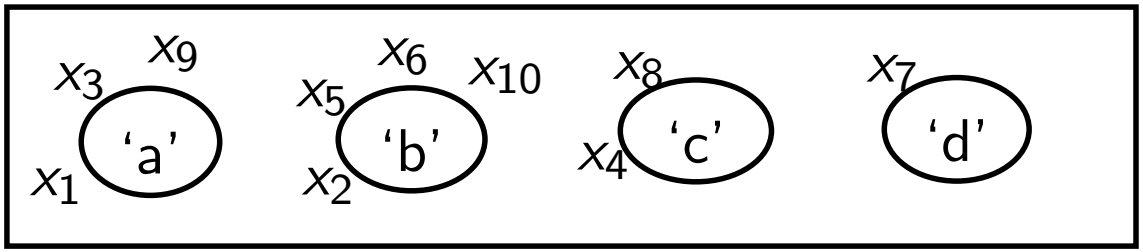
Stirling Numbers: distribution on the number of tables/species given by generalised Stirling number of second kind

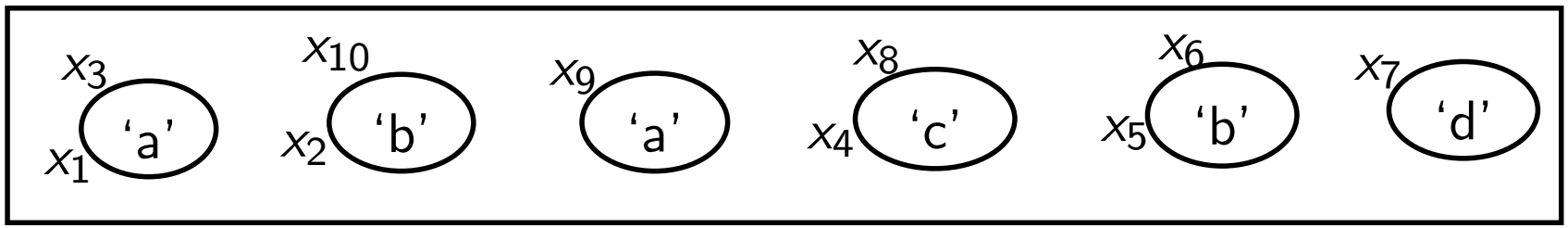
PYP and DP: are an SSM using a particular way to give the probability vector.

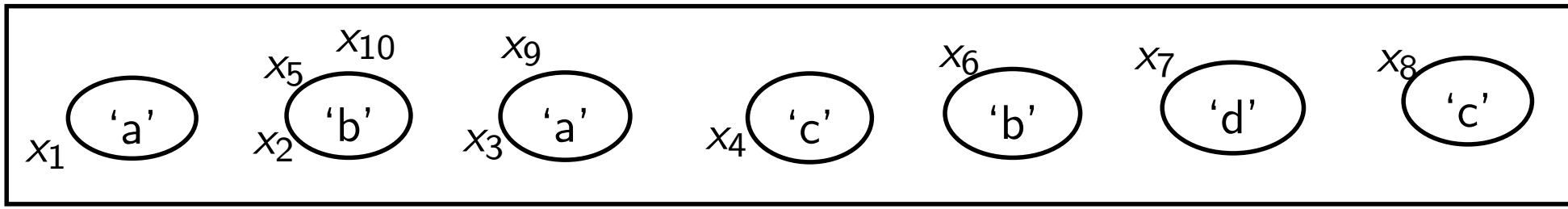
Sampling Concentration and Discount: may need to do this.

Outline

CRPs on Discrete Data







The above three table configurations all match the data stream:

$a, b, a, c, b, b, d, c, a, b$

CRPs on Discrete Data, cont.

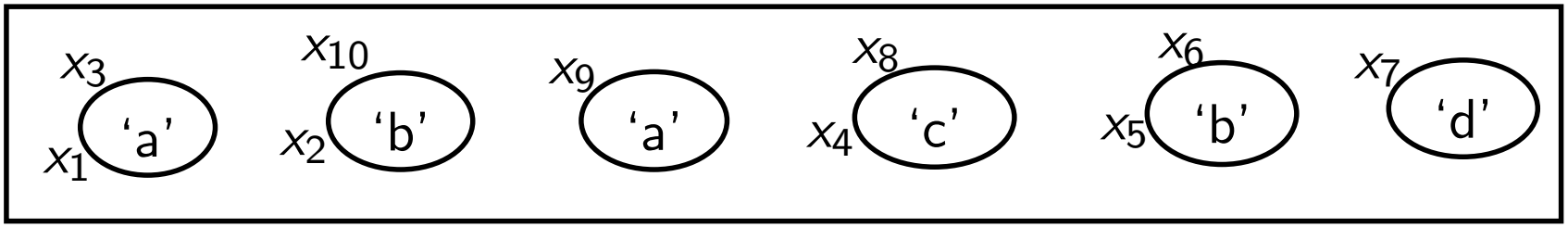
Different configurations for the data stream:

a, b, a, c, b, b, d, c, a, b

- $n_a = 3$, $n_b = 4$, $n_c = 2$, $n_d = 1$ and $N = 10$.
- So the 3 data points with 'a' could be spread over 1,2 or 3 tables!
- Thus, inference will need to know the particular configuration/assignment of data points to tables.

CRPs on Discrete Data, cont

What is the full table configuration?

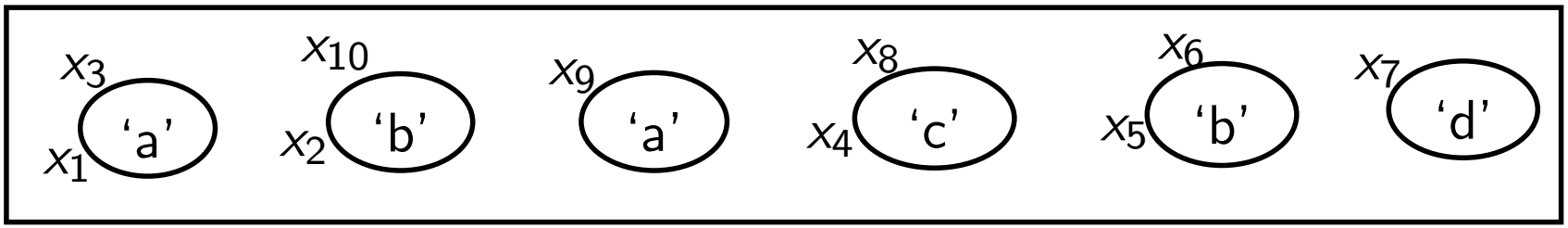


Particular configuration:

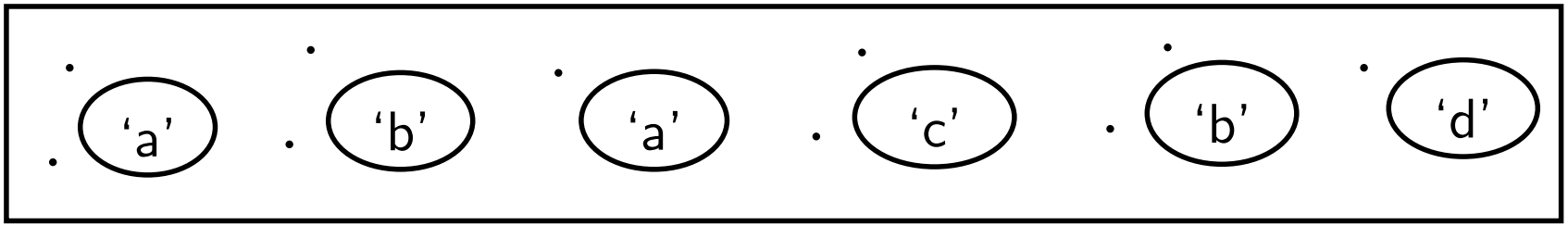
- $n_a = 3$, $n_b = 4$, $n_c = 2$, $n_d = 1$ and $N = 10$.
- How many tables for each type:
  $t_a = 2$, $t_b = 2$, $t_c = 1$, $t_d = 1$ and $T = 6$.
- How much data in each table ordered by occurence:
  $\vec{m} = (2, 2, 1, 2, 2, 1)$.
- How much data in each table ordered by type:
  $\vec{m}_a = (2, 1)$, $\vec{m}_b = (2, 2)$, $\vec{m}_c = (2)$, $\vec{m}_d = (1)$.

CRPs on Discrete Data, cont

We don't need to store the full table configuration:



We just need to store the counts in each table.



The full configuration can be reconstructed (upto some statistical variation) by uniform sampling at any stage.

Evidence of PYP for Probability Vectors

Notation:

- Tables numbered $t = 1, ..., T$. Data types numbered $k = 1, ..., K$.
- Full table configuration denoted $\mathcal{T}$.
- Count of data type k is n_k, and number of tables t_k, with constraints $t_k \leq n_k$ and $n_k > 0 \rightarrow t_k > 0$.
- Table t has data value (dish) X_t^* with m_t customers.
- Tables with data value k has vector of customers $\vec{m}_k$.

Evidence:

$$p(\vec{x}, \mathcal{T} | N, \text{PYP}, d, \alpha) = \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{t=1}^{T} ((1-d)_{m_t-1} H(X_t^*))$$

$$= \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \left(\prod_{t\,:\,X_t^*=k} (1-d)_{m_t-1} \right) H(k)^{t_k}$$

Evidence of PYP for Probability Vectors, cont.

$$p(\vec{x}, \mathcal{T}|N, \text{PYP}, d, \alpha) = \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \left(\prod_{t\,:\,X_t^*=k} (1-d)_{m_t-1} \right) H(k)^{t_k}$$

$$\begin{aligned} & p\left(\vec{x}, \vec{t}\,\middle|\, N, \text{PYP}, d, \alpha\right) \\ &= \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \sum_{\vec{m}_k \text{ s.t. } \dim(\vec{m}_k)=t_k} \left(\prod_{t=1}^{t_k} (1-d)_{m_{k,t}-1} \right) H(k)^{t_k} \\ &= \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \mathcal{S}^{n_k}_{t_k,d} H(k)^{t_k} \end{aligned}$$

The simplification uses the definition of $\mathcal{S}^n_{t,d}$.

The Pitman-Yor-Multinomial

Definition of Pitman-Yor-Multinomial

Given a discount d and concentration parameter α, a probability vector $\vec{\theta}$ of dimension L, and a count N, the Pitman-Yor-multinomial creates count vector samples $\vec{n}$ of dimension K, and auxiliary counts $\vec{t}$ (constrained by $\vec{n}$). Now $(\vec{n}, \vec{t}) \sim \text{MultPY}\left(d, \alpha, \vec{\theta}, N\right)$ denotes

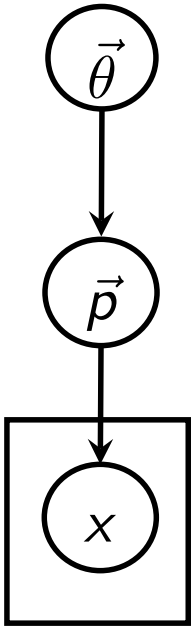
$$p\left(\vec{n}, \vec{t} \middle| N, \text{MultPY}, d, \alpha, \vec{\theta}\right) = \binom{N}{\vec{n}} \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \mathcal{S}_{t_k,d}^{n_k} \theta_k^{t_k}$$

where $T = \sum_{k=1}^{K} t_k$.

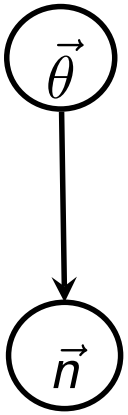
This is a form of evidence for the PYP.

The Ideal Hierarchical Component?

We want a magic distribution that looks like a multinomial likelihood in $\vec{\theta}$.



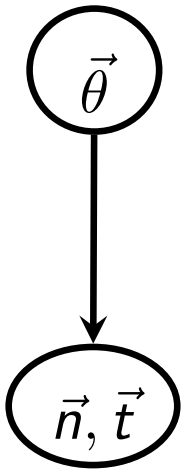
$$\begin{aligned} \vec{p} &\sim \text{Magic}\left(\alpha, \vec{\theta}\right) \\ x_n &\sim \text{Discrete}(\vec{p}) \quad \forall_n \end{aligned}$$



$$\begin{aligned} & p\left(\vec{n} \middle| \alpha, \vec{\theta}, N\right) \\ &= F_\alpha(\vec{n}) \prod_k \theta_k^{t_k} \\ &\text{where } \sum_k n_k = N \end{aligned}$$

The PYP/DP is the Magic

- The PYP/DP plays the role of the magic distribution.
- However, the exponent t_k for the θ now becomes a latent variable, so needs to be sampled as well.
- The t_k are constrainted:
  - $t_k \leq n_k$
  - $t_k > 0$ iff $n_k > 0$
- The $\vec{t}$ act like data for the next level up involving $\vec{\theta}$.



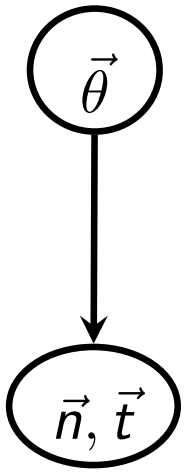
$$p\left(\vec{n}, \vec{t} \,\middle|\, d, \alpha, \vec{\theta}, N\right) = F_{d,\alpha}(\vec{n}, \vec{t}) \prod_k \theta_k^{t_k}$$

where $\sum_k n_k = N$

Interpreting the Auxiliary Counts

Interpretation: t_k is how much of the count n_k that affects the parent probability (*i.e.* $\vec{\theta}$).

- If $\vec{t} = \vec{n}$ then the sample $\vec{n}$ affects $\vec{\theta}$ 100%.
- When $n_k = 0$ then $t_k = 0$, no effect.
- If $t_k = 1$, then the sample of n_k affects $\vec{\theta}$ minimally.



$$p\left(\vec{n}, \vec{t} \,\middle|\, d, \alpha, \vec{\theta}, N\right) = F_{d,\alpha}(\vec{n}, \vec{t}) \prod_k \theta_k^{t_k}$$

where $\sum_k n_k = N$

Why We Prefer DPs and PYPs over Dirichlets!

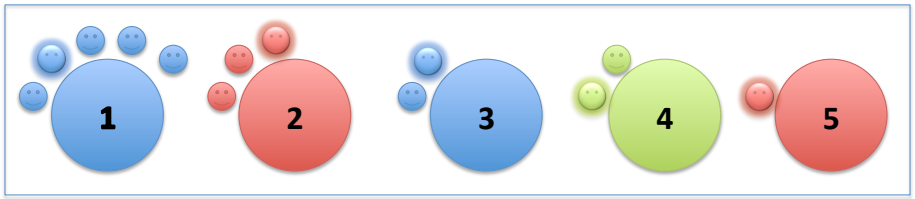
$$p\left(\vec{x}, \vec{t} \,\middle|\, N, \text{MultPY}, d, \alpha, \vec{\theta}\right) \propto \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \mathcal{S}^{n_k}_{t_k,d} {\theta_k}^{t_k} \ ,$$

$$p\left(\vec{x} \,\middle|\, N, \text{MultDir}, \alpha, \vec{\theta}\right) \propto \frac{1}{(\alpha)_N} \prod_{k=1}^{K} (\alpha\theta_k)_{n_k} \ .$$

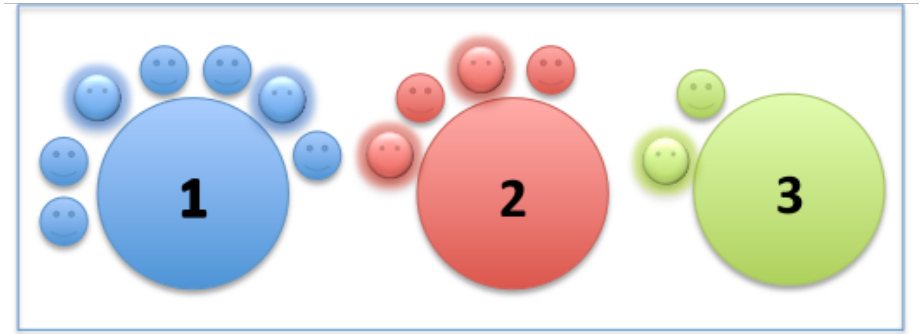
For the PYP, the θ_k just look like multinomial data, but you have to introduce a discrete latent variable $\vec{t}$.
For the Dirichlet, the θ_k are in a complex gamma function.

CRP Samplers versus MultPY Samplers

CRP sampling needs to keep track of full seating plan, such as counts per table (thus dynamic memory).



Sampling using the MultPY formula only needs to keep the number of tables. So rearrange configuration, only one table per dish and mark customers to indicate how many tables the CRP would have had.



CRP Samplers versus MultPY Samplers, cont.

CRP samplers sample configurations $\mathcal{T}$ consisting of (m_t, X_t^*) for $t = 1, ..., T$.

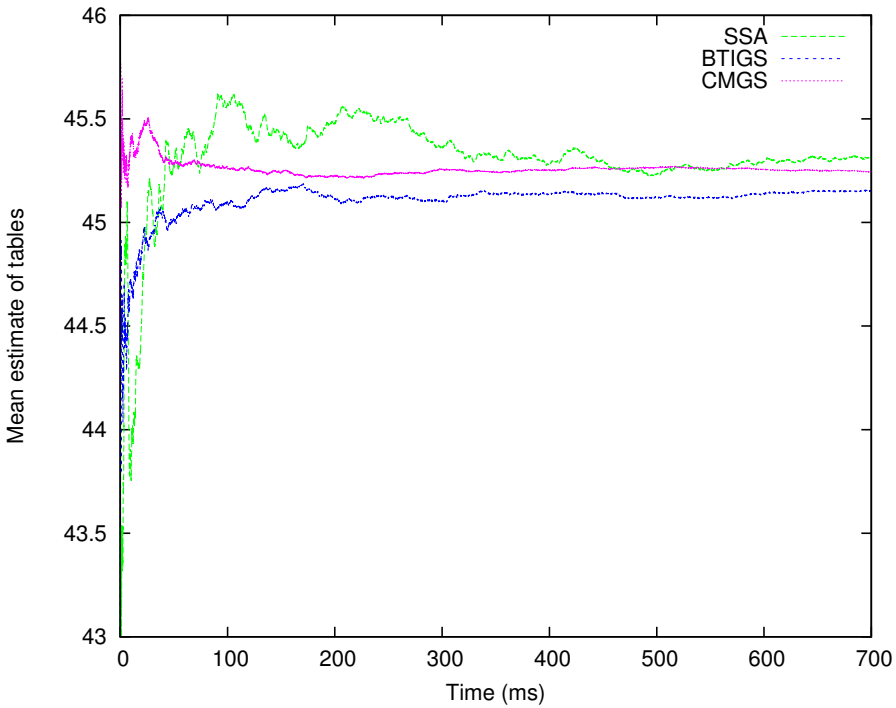
$$p(\vec{x}, \mathcal{T} | N, \text{PYP}, d, \alpha) = \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{t=1}^{T} ((1-d)_{m_t-1} H(X_t^*))$$

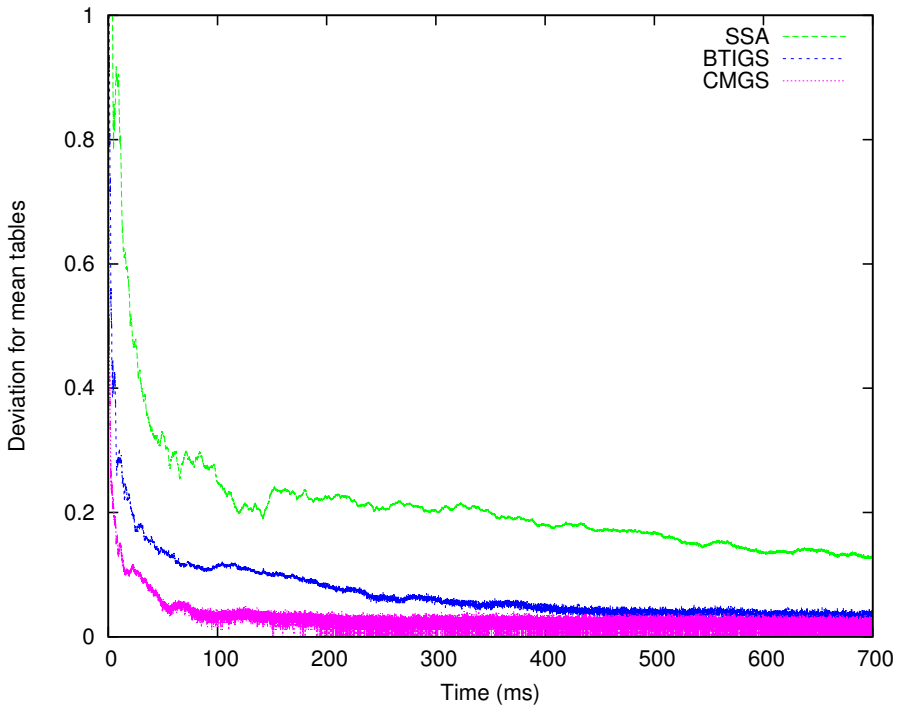
MultPY samplers sample the number of tables t_k for $k = 1, ..., K$. This is a collapsed version of a CRP sampler.

$$p\left(\vec{x}, \vec{t} \middle| N, \text{PYP}, d, \alpha\right) = \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \mathcal{S}_{t_k,d}^{n_k} H(X_t^*)^{t_k}$$

Requires O(1) access to $\mathcal{S}_{t,d}^{n}$.

Comparing Samplers for the Pitman-Yor-Multinomial



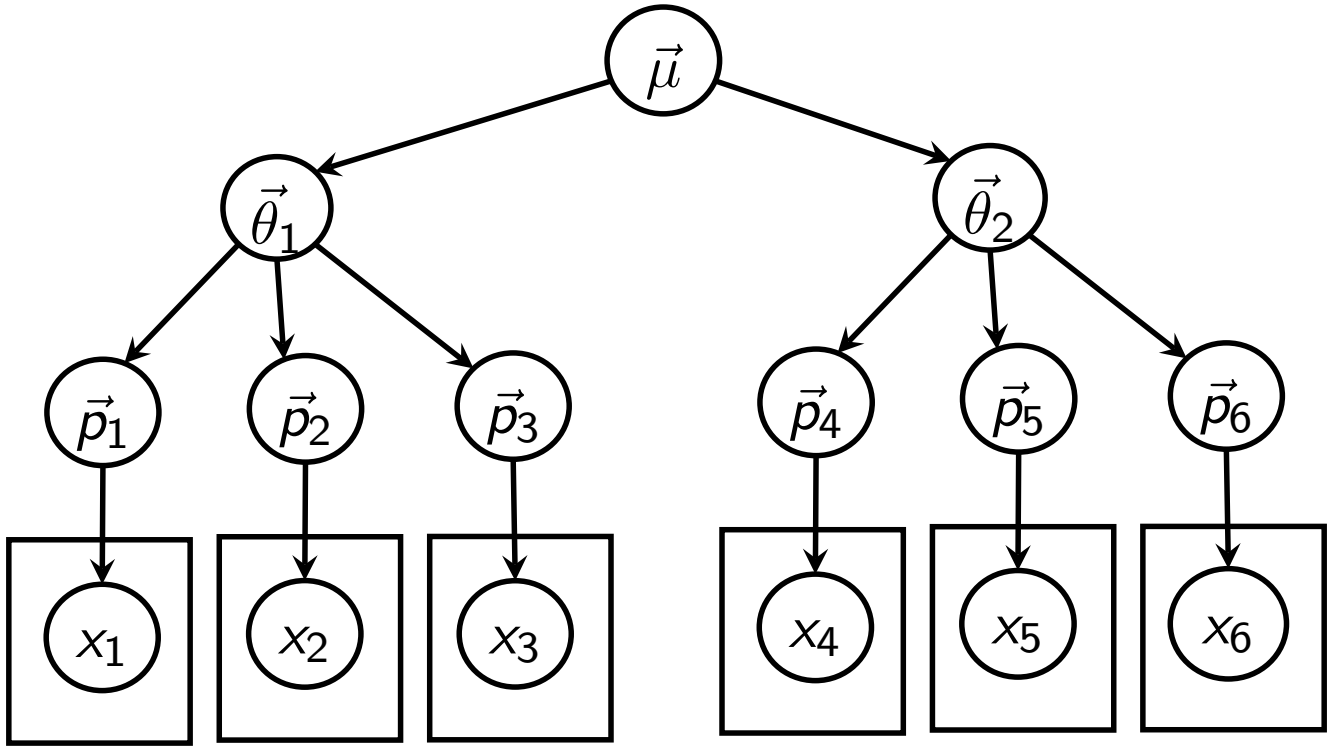


Legend: SSA = "standard CRP sampler of Teh *et al.*"
CMGS = "Gibbs sampler using MultPY posterior"

Mean estimates of the total number of tables T for one of the 20 Gibbs runs (left) and the standard deviation of the 20 mean estimates (right) with $d = 0$, $\alpha = 10$, $K = 50$ and $N = 500$.

Outline

A Simple N-gram Style Model



$$p(\vec{\mu})p\left(\vec{\theta}_1 \middle| \vec{\mu}\right) p\left(\vec{\theta}_2 \middle| \vec{\mu}\right)$$

$$p\left(\vec{p}_1 \middle| \vec{\theta}_1\right) p\left(\vec{p}_2 \middle| \vec{\theta}_1\right) p\left(\vec{p}_3 \middle| \vec{\theta}_1\right) p\left(\vec{p}_4 \middle| \vec{\theta}_2\right) p\left(\vec{p}_5 \middle| \vec{\theta}_2\right) p\left(\vec{p}_6 \middle| \vec{\theta}_2\right)$$

$$\prod_l p_{1,l}^{n_{1,l}} \prod_l p_{2,l}^{n_{2,l}} \prod_l p_{3,l}^{n_{3,l}} \prod_l p_{4,l}^{n_{4,l}} \prod_l p_{5,l}^{n_{5,l}} \prod_l p_{6,l}^{n_{6,l}}$$

Using the Evidence Formula

We will repeatedly apply the evidence formula

$$\begin{aligned} p\left(\vec{x}, \vec{t} \middle| N, \mathrm{DP}, \alpha\right) &= \frac{\alpha^T}{(\alpha)_N} \prod_{k=1}^{K} \mathcal{S}^{n_k}_{t_k,d} H(k)^{t_k} \\ &= F_\alpha(\vec{n}, \vec{t}) \prod_{k=1}^{K} H(k)^{t_k} \end{aligned}$$

to marginalise out all the probability vectors.

Apply Evidence Formula to Bottom Level

Start with the full posterior:

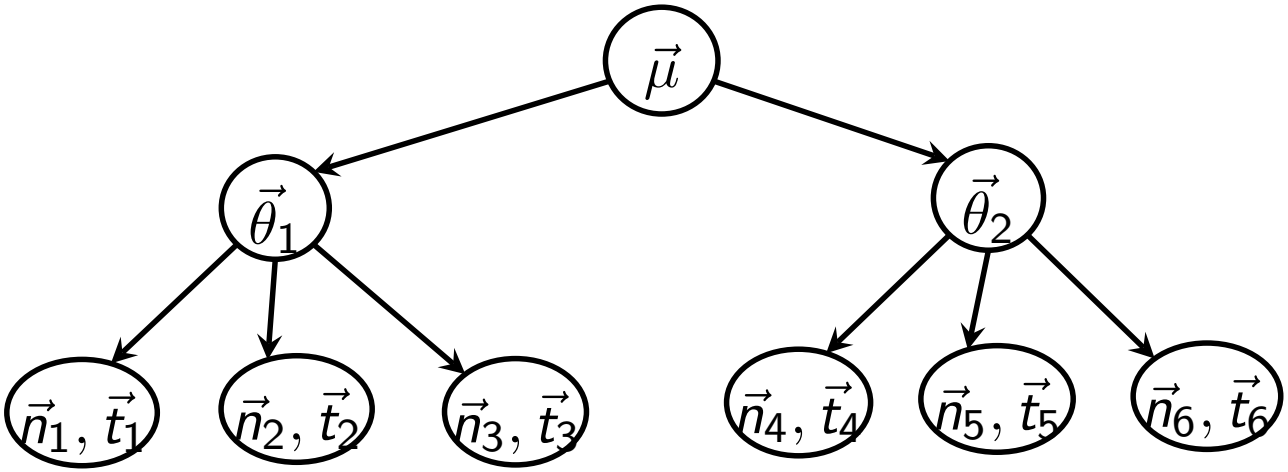
$$p(\vec{\mu})p\left(\vec{\theta}_1 \middle| \vec{\mu}\right) p\left(\vec{\theta}_2 \middle| \vec{\mu}\right)$$
$$p\left(\vec{p}_1 \middle| \vec{\theta}_1\right) p\left(\vec{p}_2 \middle| \vec{\theta}_1\right) p\left(\vec{p}_3 \middle| \vec{\theta}_1\right) p\left(\vec{p}_4 \middle| \vec{\theta}_2\right) p\left(\vec{p}_5 \middle| \vec{\theta}_2\right) p\left(\vec{p}_6 \middle| \vec{\theta}_2\right)$$
$$\prod_l p_{1,l}^{n_{1,l}} \prod_l p_{2,l}^{n_{2,l}} \prod_l p_{3,l}^{n_{3,l}} \prod_l p_{4,l}^{n_{4,l}} \prod_l p_{5,l}^{n_{5,l}} \prod_l p_{6,l}^{n_{6,l}} \ .$$

Marginalise out each $\vec{p}_k$ but introducing new auxiliaries $\vec{t}_k$

$$p(\vec{\mu})p\left(\vec{\theta}_1 \middle| \vec{\mu}\right) p\left(\vec{\theta}_2 \middle| \vec{\mu}\right)$$
$$F_\alpha(\vec{n}_1, \vec{t}_1)F_\alpha(\vec{n}_2, \vec{t}_2)F_\alpha(\vec{n}_3, \vec{t}_3) \prod_l \theta_{1,l}^{t_{1,l}+t_{2,l}+t_{3,l}}$$
$$F_\alpha(\vec{n}_4, \vec{t}_4)F_\alpha(\vec{n}_5, \vec{t}_5)F_\alpha(\vec{n}_6, \vec{t}_6) \prod_l \theta_{2,l}^{t_{4,l}+t_{5,l}+t_{6,l}} \ .$$

Thus $\vec{t}_1 + \vec{t}_2 + \vec{t}_3$ looks like data for $\vec{\theta}_1$ and $\vec{t}_4 + \vec{t}_5 + \vec{t}_6$ looks like data for $\vec{\theta}_2$

Apply Evidence Formula, cont.



Terms left in $\vec{n}_k$ and $\vec{t}_k$, and passing up

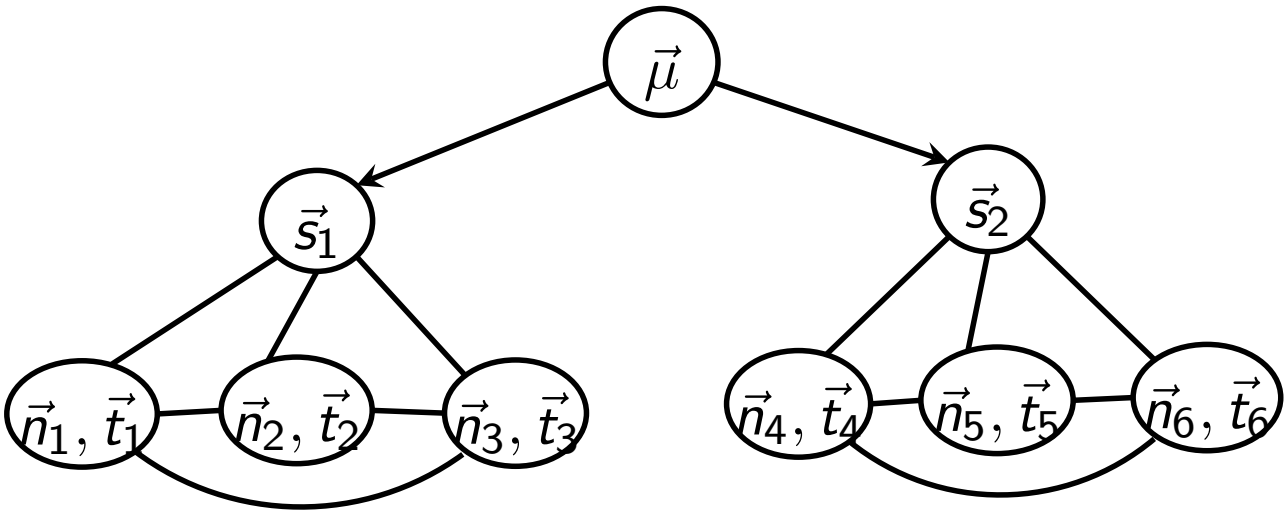
$$\prod_l \theta_{1,l}^{t_{1,l}+t_{2,l}+t_{3,l}} \prod_l \theta_{2,l}^{t_{4,l}+t_{5,l}+t_{6,l}} ,$$

as pseudo-data to the prior on $\vec{\theta}_1$ and $\vec{\theta}_2$.

Apply Evidence Formula, cont.

Repeat the same trick up a level; marginalising out $\vec{\theta}_1$ and $\vec{\theta}_1$ but introducing new auxiliaries $\vec{s}_1$ and $\vec{s}_2$

$$p(\vec{\mu})F_\alpha(\vec{t}_1+\vec{t}_2+\vec{t}_3,\vec{s}_1)F_\alpha(\vec{t}_4+\vec{t}_5+\vec{t}_6,\vec{s}_2)\prod_l \mu_l^{s_{1,l}+s_{2,l}}$$
$$F_\alpha(\vec{n}_1,\vec{t}_1)F_\alpha(\vec{n}_2,\vec{t}_2)F_\alpha(\vec{n}_3,\vec{t}_3)F_\alpha(\vec{n}_4,\vec{t}_4)F_\alpha(\vec{n}_5,\vec{t}_5)F_\alpha(\vec{n}_6,\vec{t}_6)\ .$$



Again left with pseudo-data to the prior on $\vec{\mu}$.

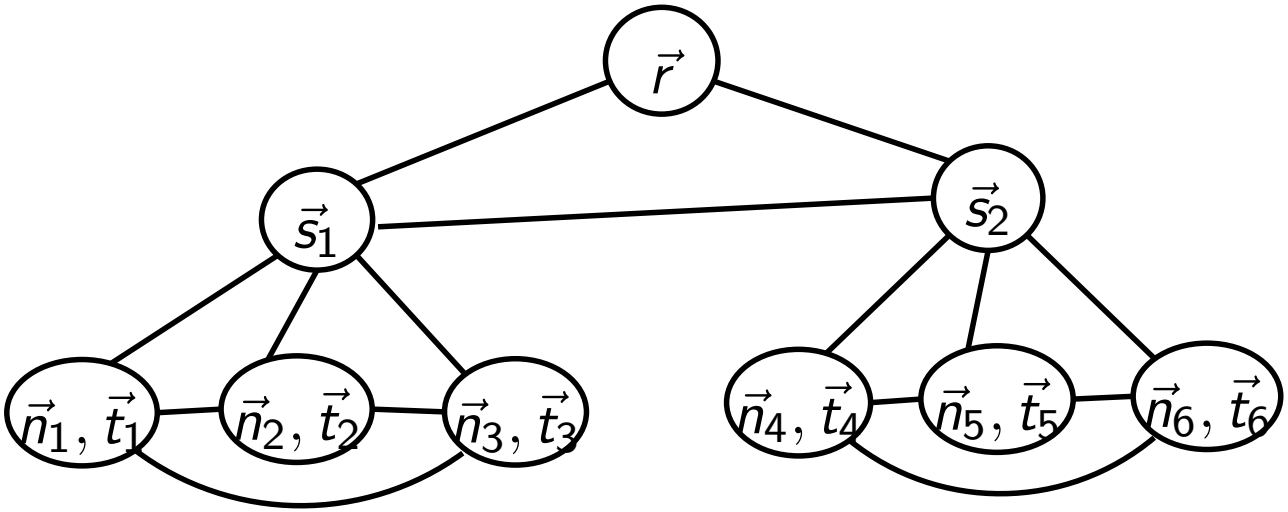
Apply Evidence Formula, cont.

Finally repeat at the top level with new auxiliary $\vec{r}$

$$F_\alpha(\vec{s}_1 + \vec{s}_2, \vec{r})F_\alpha(\vec{t}_1 + \vec{t}_2 + \vec{t}_3, \vec{s}_1)F_\alpha(\vec{t}_4 + \vec{t}_5 + \vec{t}_6, \vec{s}_2)$$
$$F_\alpha(\vec{n}_1, \vec{t}_1)F_\alpha(\vec{n}_2, \vec{t}_2)F_\alpha(\vec{n}_3, \vec{t}_3)F_\alpha(\vec{n}_4, \vec{t}_4)F_\alpha(\vec{n}_5, \vec{t}_5)F_\alpha(\vec{n}_6, \vec{t}_6)$$

where

- $\vec{n}_1$, $\vec{n}_2$,... are the data at the leaf nodes, $\vec{t}_1$, $\vec{t}_2$,... their auxiliary counts
- $\vec{s}_1$ are auxiliary counts constrained by $\vec{t}_1 + \vec{t}_2 + \vec{t}_3$,
- $\vec{s}_2$ are auxiliary counts constrained by $\vec{t}_4 + \vec{t}_5 + \vec{t}_6$,
- $\vec{r}$ are auxiliary counts constrained by $\vec{s}_1 + \vec{s}_2$,



The Worked N-gram Style Model

Original posterior in the form:

$$p(\vec{\mu})p\left(\vec{\theta}_1 \middle| \vec{\mu}\right) p\left(\vec{\theta}_2 \middle| \vec{\mu}\right)$$
$$p\left(\vec{p}_1 \middle| \vec{\theta}_1\right) p\left(\vec{p}_2 \middle| \vec{\theta}_1\right) p\left(\vec{p}_3 \middle| \vec{\theta}_1\right) p\left(\vec{p}_4 \middle| \vec{\theta}_2\right) p\left(\vec{p}_5 \middle| \vec{\theta}_2\right) p\left(\vec{p}_6 \middle| \vec{\theta}_2\right)$$
$$\prod_l p_{1,l}^{n_{1,l}} \prod_l p_{2,l}^{n_{2,l}} \prod_l p_{3,l}^{n_{3,l}} \prod_l p_{4,l}^{n_{4,l}} \prod_l p_{5,l}^{n_{5,l}} \prod_l p_{6,l}^{n_{6,l}}$$

Collapsed posterior in the form:

$$F_\alpha(\vec{s}_1 + \vec{s}_2, \vec{r})F_\alpha(\vec{t}_1 + \vec{t}_2 + \vec{t}_3, \vec{s}_1)F_\alpha(\vec{t}_4 + \vec{t}_5 + \vec{t}_6, \vec{s}_2)$$
$$F_\alpha(\vec{n}_1, \vec{t}_1)F_\alpha(\vec{n}_2, \vec{t}_2)F_\alpha(\vec{n}_3, \vec{t}_3)F_\alpha(\vec{n}_4, \vec{t}_4)F_\alpha(\vec{n}_5, \vec{t}_5)F_\alpha(\vec{n}_6, \vec{t}_6)$$

where

- $\vec{n}_1$, $\vec{n}_2$,... are the data at the leaf nodes, $\vec{t}_1$, $\vec{t}_2$,... their auxiliary counts
- $\vec{s}_1$ are auxiliary counts constrained by $\vec{t}_1 + \vec{t}_2 + \vec{t}_3$,
- $\vec{s}_2$ are auxiliary counts constrained by $\vec{t}_4 + \vec{t}_5 + \vec{t}_6$,
- $\vec{r}$ are auxiliary counts constrained by $\vec{s}_1 + \vec{s}_2$,

The Worked N-gram Style Model, cont.

Note the probabilities are then estimated from the auxiliary counts during MCMC. This is the standard recursive CRP formula.

$$\begin{aligned}
\widehat{\vec{\mu}} &= \frac{\vec{s}_1 + \vec{s}_2}{S_1 + S_2 + \alpha} + \frac{\alpha}{S_1 + S_2 + \alpha}\left(\frac{\vec{r}}{R + \alpha} + \frac{R}{R + \alpha}\frac{1}{L}\right) \\
\widehat{\vec{\theta}}_1 &= \frac{\vec{t}_1 + \vec{t}_2 + \vec{t}_3}{T_1 + T_2 + T_3 + \alpha} + \frac{\alpha}{T_1 + T_2 + T_3 + \alpha}\widehat{\vec{\mu}} \\
\widehat{\vec{p}}_1 &= \frac{\vec{n}_1}{N_1 + \alpha} + \frac{\alpha}{N_1 + \alpha}\widehat{\vec{\theta}}_1
\end{aligned}$$

Note in practice:

- the α is varied at every level of the tree and sampled as well,
- the PYP is used instead because words are often Zipfian

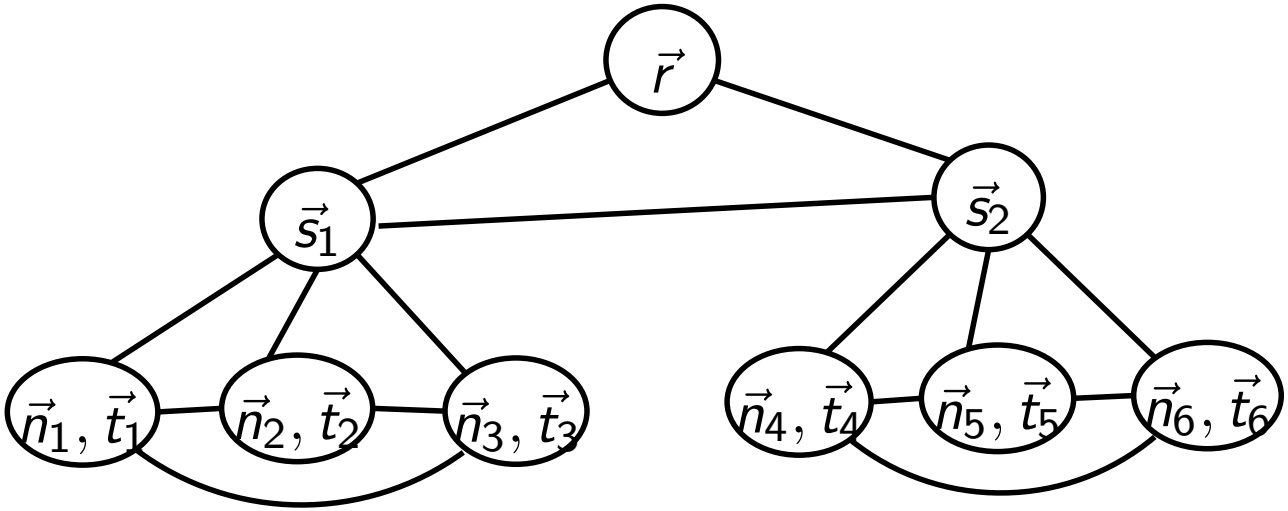
The Worked N-gram Style Model, cont.

What have we achieved:

- Bottom level probabilities $(\vec{p}_1, \vec{p}_2, \ldots)$ marginalised away.
- Each non-leaf probability vector $(\vec{\mu}, \vec{\theta}_1, \ldots)$ replaced by corresponding constrained auxiliary count vector $(\vec{r}, \vec{s}_1, \ldots)$ as psuedo-data.
- The auxiliary counts correspond to how much of the counts get inherited up the hierarchy.
- This allows a collapsed sampler in a discrete (versus continuous) space.

MCMC Problem Specification for N-grams

Build a Gibbs/MCMC sampler for:



$$(\vec{\mu}) \quad \cdots \quad \frac{\alpha^R}{(\alpha)_{S_1+S_2}} \prod_{k=1}^{K} \left(\mathcal{S}^{s_{1,k}+s_{2,k}}_{r_k,0} \frac{1}{K^{r_k}} \right)$$

$$(\vec{\theta}_1, \vec{\theta}_2) \quad \cdots \quad \frac{\alpha^{S_1}}{(\alpha)_{T_1+T_2+T_3}} \prod_{k=1}^{K} \mathcal{S}^{t_{1,k}+t_{2,k}+t_{3,k}}_{s_{1,k},0} \frac{\alpha^{S_2}}{(\alpha)_{T_4+T_5+T_6}} \prod_{k=1}^{K} \mathcal{S}^{t_{4,k}+t_{5,k}+t_{6,k}}_{s_{2,k},0}$$

$$(\forall_k \vec{p}_k) \quad \cdots \quad \prod_{l=1}^{6} \left(\prod_{k=1}^{K} \mathcal{S}^{n_{l,k}}_{t_{l,k},0} \right)$$

Sampling Ideas

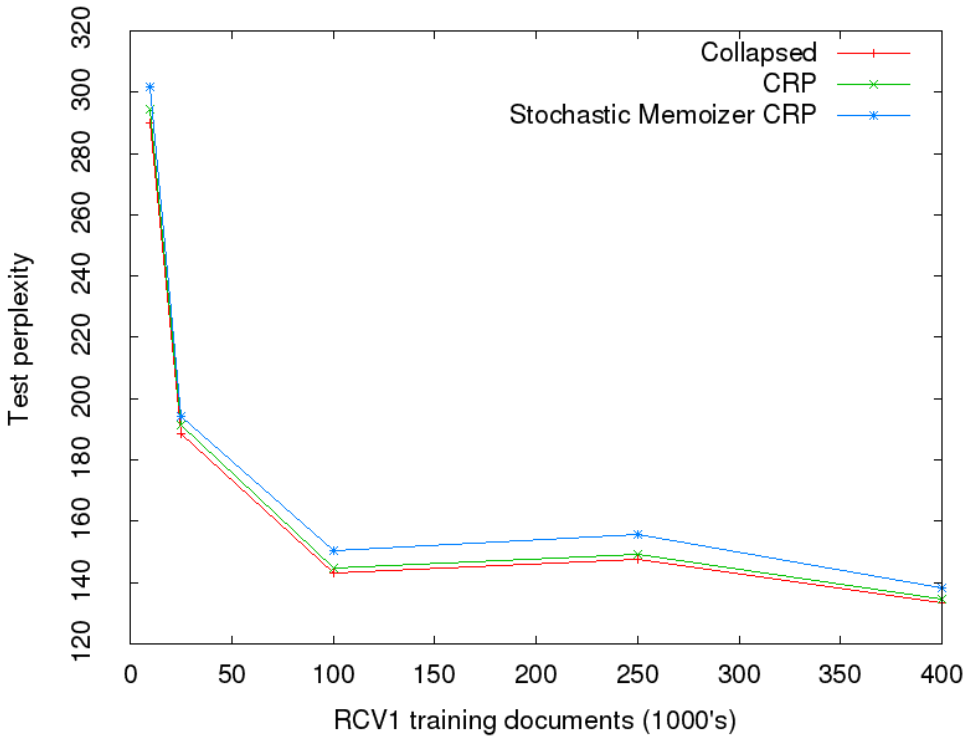
Consider the term in $s_{1,k}$ where $s_{1,k} \leq t_{1,k} + t_{2,k} + t_{3,k}$.

$$\frac{1}{(\alpha)_{S_1'+S_2+s_{1,k}}} \mathcal{S}_{r_k,0}^{s_{1,k}+s_{2,k}} \alpha^{S_1'+s_{1,k}} \mathcal{S}_{s_{1,k},0}^{t_{1,k}+t_{2,k}+t_{3,k}}$$

- Gibbs by sampling $s_{1,k}$ proportional to this for all $1 \leq s_{1,k} \leq t_{1,k} + t_{2,k} + t_{3,k}$.
- Approximate Gibbs by sampling $s_{1,k}$ proportional to this for $s_{1,k}$ in a window of size 21 around the current: $\max(1, s_{1,k} - 10) \leq s_{1,k} \leq \min(s_{1,k} + 10, t_{1,k} + t_{2,k} + t_{3,k})$.
- Metropolis-Hastings by sampling $s_{1,k}$ proportional to this for $s_{1,k}$ in a window of size 3 or 5 or 11.

Note: have used the second in implementations.

Some Results on RCV1 with 5-grams



Collapsed is the method here using PYPs with both discount and concentration sampled level-wise.

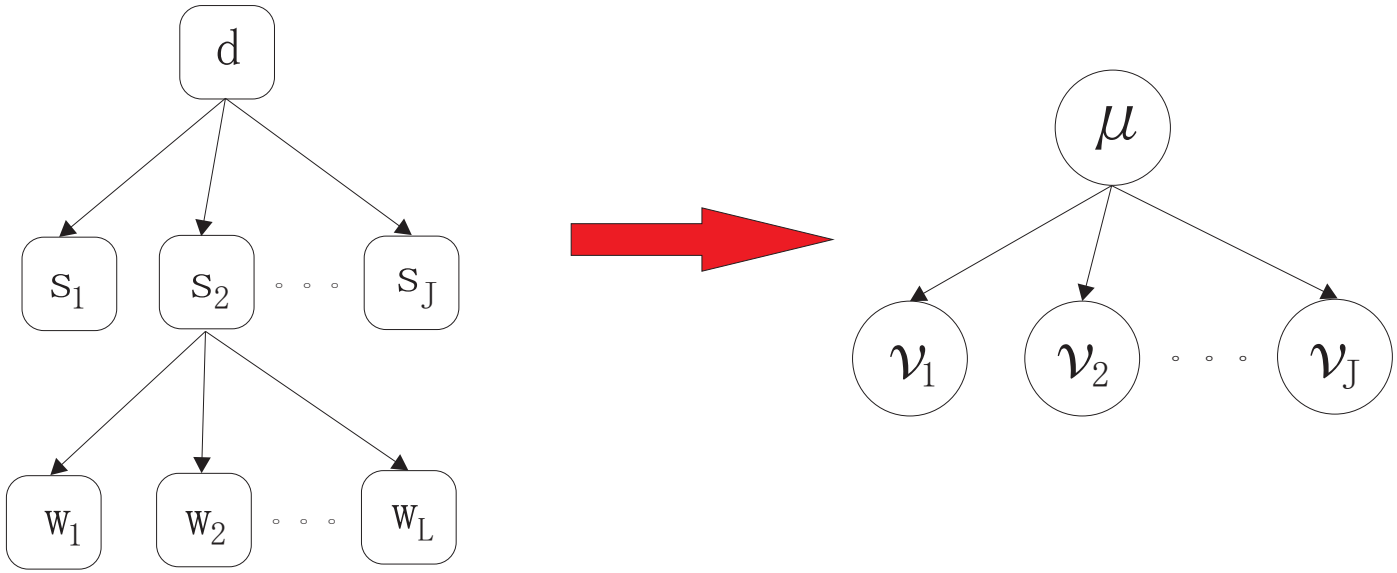
CRP is the CRP method of Teh (ACL 2006) with both discount and concentration sampled level-wise.

Memoizer fixes the CRP parameters to that Wood *et al.* (ICML 2009).

- Used Reuters RCV1 collection with 400k documents (about 190M words), and following 5k for test.
- Gave methods equal time.
- `Collapsed` and `CRP` exhibit similar convergence.
- `Collapsed` requires no dynamic memory so takes about 1/2 of the space.
- `Collapsed` improves with 10-grams on full RCV1.

Outline

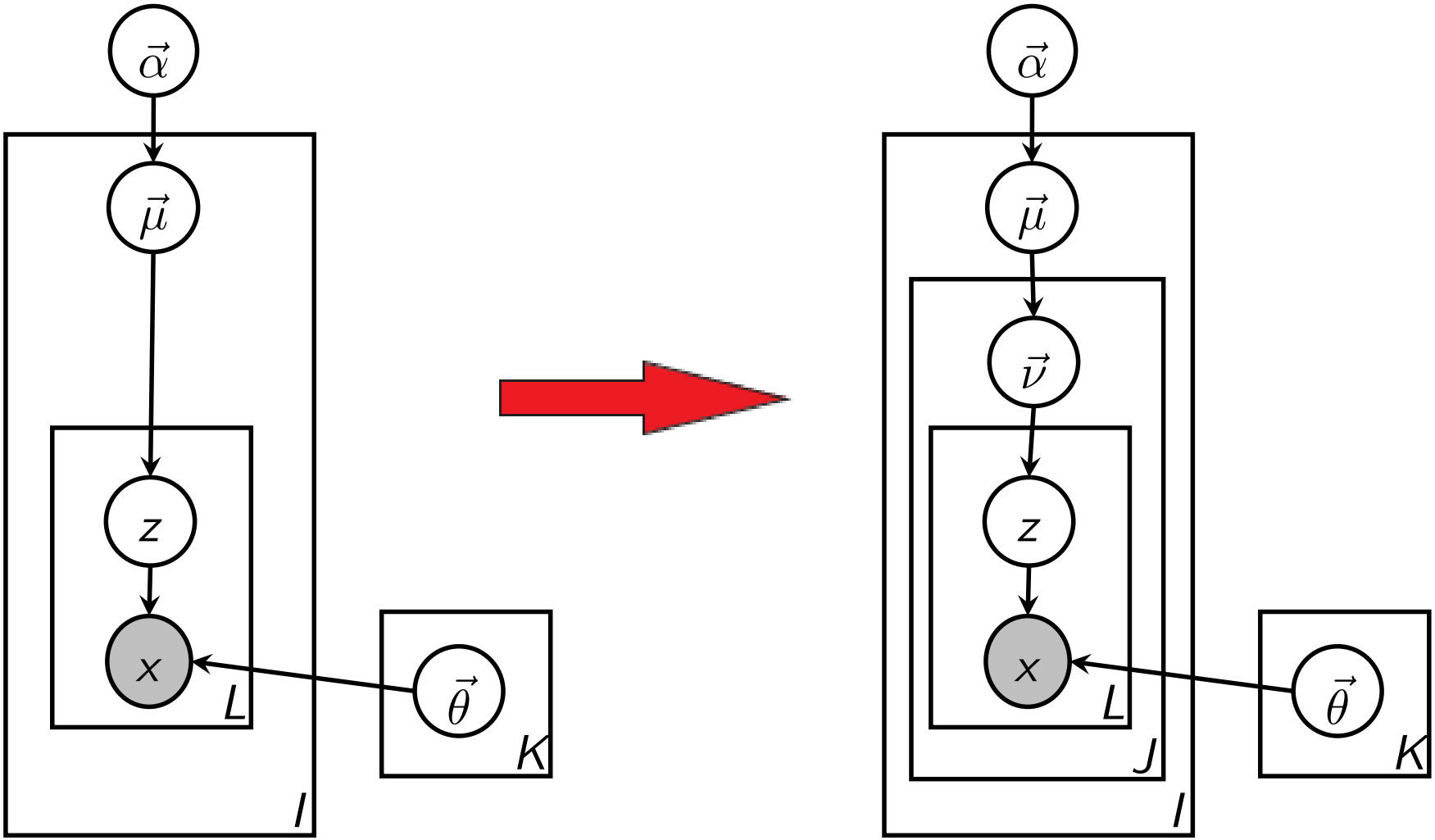
Structured Documents



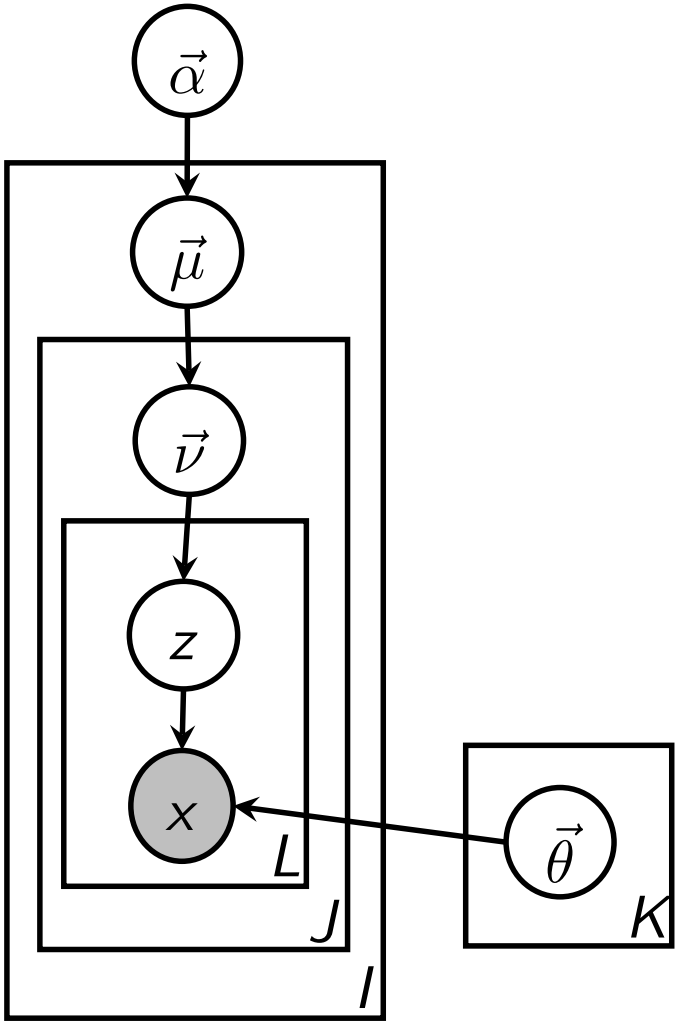
- A document contains sections which contains words.
- This implies the graphical model between the topics of the document and its sections.

Structured Topic Model (STM)

We add this "structure" to the standard topic model.



Structured Topic Model, cont.



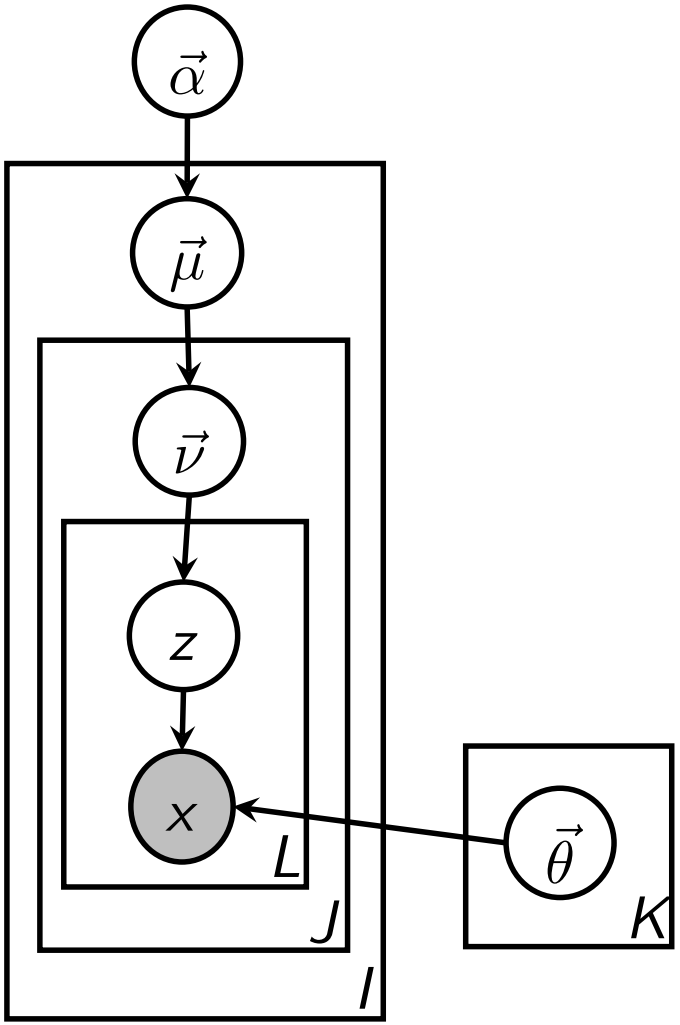
$$
\begin{aligned}
\vec{\theta}_k &\sim \text{Dirichlet}_V(\vec{\gamma}) \quad \forall_{k=1}^{K} , \\
\vec{\mu}_i &\sim \text{Dirichlet}_K(\vec{\alpha}) \quad \forall_{i=1}^{I} , \\
\vec{\nu}_{i,j} &\sim \text{Dirichlet}_K(\beta, \vec{\mu}_i) \quad \forall_{i=1}^{I} \forall_{j=1}^{J_i} , \\
z_{i,j,l} &\sim \text{Discrete}(\vec{\nu}_{i,j}) \quad \forall_{i=1}^{I} \forall_{j=1}^{J_i} \forall_{l=1}^{L_{i,j}} , \\
x_{i,j,l} &\sim \text{Discrete}\left(\vec{\theta}_{z_{i,j,l}}\right) \quad \forall_{i=1}^{I} \forall_{j=1}^{J_i} \forall_{l=1}^{L_{i,j}} .
\end{aligned}
$$

where

$$
\begin{aligned}
K &:= \# \text{ topics}, \\
V &:= \# \text{ words}, \\
I &:= \# \text{ documents}, \\
J_i &:= \# \text{ segments in doc } i, \\
L_{i,j} &:= \# \text{ words in seg } j \text{ of doc } i
\end{aligned}
$$

Extending LDA, Strategy

Structure Topic Model



Consider the collapsed LDA posterior:

$$\prod_{i=1}^{I} \frac{\text{Beta}_K\left(\vec{\alpha}+\vec{m}_i\right)}{\text{Beta}_K\left(\vec{\alpha}\right)} \prod_{k=1}^{K} \frac{\text{Beta}_V\left(\vec{\gamma}+\vec{n}_k\right)}{\text{Beta}_V\left(\vec{\gamma}\right)}$$

We can extend LDA in all sorts of ways by replacing the Dirichlet-multinomial parts with Dirichlet-multinomial processes or Pitman-Yor-multinomial.

- expanding vocabulary;
- expanding topics (called HDP-LDA);
- also, Structured topic models.

Structured Topic Model Posterior

Full posterior:

$$= \prod_{k=1}^{K} p(\vec{\theta}_k|\vec{\gamma}) \prod_{i=1}^{I} p(\vec{\mu}_i|\vec{\alpha}) \prod_{i,j=1}^{I,J_i} p(\vec{\nu}_{i,j}|\beta,\vec{\mu}_i) \prod_{i,j,l=1}^{I,J_i,L_{i,j}} \nu_{i,j,z_{i,j,l}} \theta_{z_{i,j,l},x_{i,j,l}}$$

$$= \overbrace{\prod_{i=1}^{I} p(\vec{\mu}_i|\vec{\alpha})}^{\text{terms in } \vec{\mu}_i} \overbrace{\prod_{i,j=1}^{I,J_i} p(\vec{\nu}_{i,j}|\beta,\vec{\mu}_i) \prod_{i,j,k=1}^{I,J_i,K} \nu_{i,j,k}^{m_{i,j,k}}}^{\text{terms in } \vec{\mu}_i + \vec{\nu}_{i,j}} \overbrace{\prod_{k=1}^{K} p(\vec{\theta}_k|\vec{\gamma}) \prod_{k,v=1}^{K,V} \theta_{k,v}^{n_{k,v}}}^{\text{terms in } \vec{\theta}_k}$$

$$\Rightarrow \overbrace{\prod_{i=1}^{I} F_{\alpha_0}(\vec{t}_{i,\cdot},\vec{s}_i) \prod_{i,k=1}^{I,K} \left(\frac{\alpha_k}{\alpha_0}\right)^{s_{i,k}}}^{\text{marginalising } \vec{\mu}_i} \overbrace{\prod_{i,j=1}^{I,J_i} F_{\beta}(\vec{m}_{i,j},\vec{t}_{i,j})}^{\text{marginalising } \vec{\nu}_{i,j}} \overbrace{\prod_{k=1}^{K} \frac{\text{Beta}_V(\vec{\gamma}+\vec{n}_k}{\text{Beta}_V(\vec{\gamma})}}^{\text{marginalising } \vec{\theta}_k}$$

Marginalise using the same methods as before.

Structured Topic Model Posterior, cont.

$$\overbrace{\prod_{i=1}^{I} \frac{1}{(\alpha_0)_{T_{i,\cdot}}} \prod_{i,k=1}^{I,K} \mathcal{S}^{t_{i,\cdot,k}}_{s_{i,k},0} \alpha_k^{s_{i,k}}}^{\text{marginalising } \vec{\mu}_i} \overbrace{\prod_{i,j=1}^{I,J_i} \frac{\beta^{T_{i,j}}}{(\beta)_{M_{i,j}}} \prod_{i,j,k=1}^{I,J_i,K} \mathcal{S}^{m_{i,j,k}}_{t_{i,j,k},0}}^{\text{marginalising } \vec{\nu}_{i,j}} \overbrace{\prod_{k=1}^{K} \frac{\text{Beta}_V(\vec{\gamma} + \vec{n}_k)}{\text{Beta}_V(\vec{\gamma})}}^{\text{marginalising } \vec{\theta}_k}$$

with statistics

$\vec{m}_{i,j}$:= dim(K) data counts of topics for seg j in doc i, given by $m_{i,j,k} = \sum_{l=1}^{L_{i,j}} 1_{z_{i,j,l}=k}$

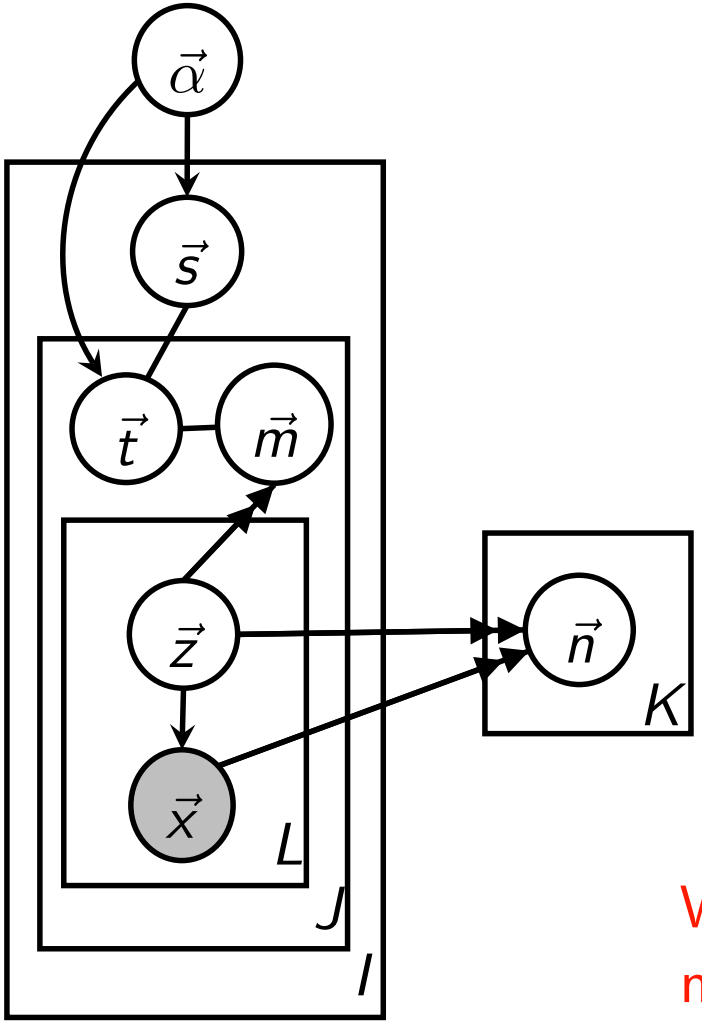
$\vec{n}_k$:= dim(V) data counts of words for topic k, given by $n_{k,v} = \sum_{i,j,l=1}^{I,J_i,L_{i,j}} 1_{z_{i,j,l}=k} 1_{x_{i,j,l}=v}$

$\vec{t}_{i,j}$:= dim(K) auxiliary counts for $\vec{\mu}_i$ from seg j in doc i, constrained by $\vec{m}_{i,j}$,

$\vec{s}_i$:= dim(K) auxiliary counts for $\vec{\alpha}$ from doc i, constrained by $\vec{t}_{i,\cdot}$.

and totals: $\vec{t}_{i,\cdot} = \sum_{j=1}^{J_i} \vec{t}_{i,j}$, $T_{i,j} = \sum_{k=1}^{K} t_{i,j,k}$, $M_{i,j} = \sum_{k=1}^{K} m_{i,j,k}$.

Structured Topic Model Posterior, cont.



$\vec{m}_{i,j}$:= $\dim(K)$ data counts of topics for seg j in doc i,
given by $m_{i,j,k} = \sum_{l=1}^{L_{i,j}} 1_{z_{i,j,l}=k}$

$\vec{n}_k$:= $\dim(V)$ data counts of words for topic k, given by
$n_{k,v} = \sum_{i,j,l=1}^{I,J_i,L_{i,j}} 1_{z_{i,j,l}=k} 1_{x_{i,j,l}=v}$

$\vec{t}_{i,j}$:= $\dim(K)$ auxiliary counts for $\vec{\mu}_i$ from seg j in doc i,
constrained by $\vec{m}_{i,j}$,

$\vec{s}_i$:= $\dim(K)$ auxiliary counts for $\vec{\alpha}$ from doc i,
constrained by $\vec{t}_{i,\cdot} = \sum_j \vec{t}_{i,j}$

We need to sample the topics $z_{i,j,l}$, all the while maintaining the counts $\vec{n}_k$ and $\vec{m}_{i,j}$, and concurrently resampling $\vec{t}_{i,j}$ and $\vec{s}_i$.

Sampling Notes

The key variables being sampled and their relevant terms are:

$$\overbrace{\mathcal{S}^{m_{i,j,k}}_{t_{i,j,k},0}\frac{\gamma_{x_{i,j,l}}+n_{k,x_{i,j,l}}}{\sum_v(\gamma_v+n_{k,v})}}^{z_{i,j,l}=k}\quad\overbrace{\frac{\beta^{t_{i,j,k}}}{(\alpha_0)_{T_{i,\cdot}}}\mathcal{S}^{t_{i,\cdot,k}}_{s_{i,k},0}\mathcal{S}^{m_{i,j,k}}_{t_{i,j,k},0}}^{t_{i,j,k}}\quad\overbrace{\beta^{s_{i,k}}\mathcal{S}^{t_{i,\cdot,k}}_{s_{i,k},0}}^{s_{i,k}}$$

- Note $t_{i,j,k}$ is correlated with $m_{i,j,k}$, and $s_{i,k}$ is correlated with $t_{i,j,k}$.
- Option is to sample sequentially $z_{i,j,l}$, $t_{i,j,k}$ and $s_{i,k}$ (*i.e.*, sweeping up the hierarchy) in turn,
  - can be expensive if full sampling ranges done, *e.g.*, $s_{i,k} : 1 \leq s_{i,k} \leq t_{i,\cdot,k}$.
- In practice, works OK, but is not great: mixing is poor!

Summary: Simple PYP Sampling

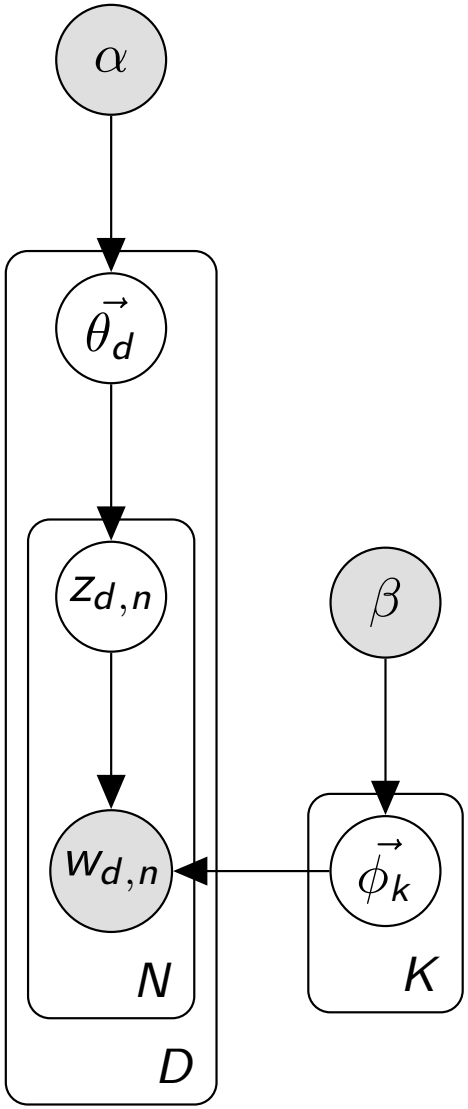
- probabilities in each PYP hierarchy are marginalised out from the bottom up.
- simple sampling strategy $\equiv$ sample the numbers of tables vectors $(\vec{t})$
- with the n-gram, the leaves of the PYP hierarchy are observed, and a simple sampling strategy works well
  - Teh tried this (2006a, p16) but says "it is expensive to compute the generalized Stirling numbers."
- with unsupervised models generally, like STM, the leaves of the PYP hierarchy are unobserved and the simple sampling strategy gives poor mixing
- on more complex models, not clear simple sampling strategy is any better than hierarchical CRP sampling

Outline

Previous Work on Non-parametric Topic Models

- "Hierarchical Dirichlet Processes," Teh, Jordan, Beal, Blei 2006.
- "Rethinking LDA: Why priors matter," Wallach, Mimno, McCallum, 2009.
- "Accounting for burstiness in topic models," Doyle and Elkan 2009.
- "Topic models with power-law using Pitman-Yor process," Sato and Nakagawa 2010
- Sampling table configurations for the hierarchical Poisson-Dirichlet process," Chen, Du and Buntine 2011.
- "Practical collapsed variational Bayes inference for hierarchical Dirichlet process," Sato, Kurihara, and Nakagawa 2012.
- "Truly nonparametric online variational inference for hierarchical Dirichlet processes," Bryant and Sudderth 2012.
- "Stochastic Variational Inference," Hoffman, Blei, Wang and Paisley 2013.
- "Latent IBP compound Dirichlet Allocation," Archambeau, Lakshminarayanan, Bouchard 2014.

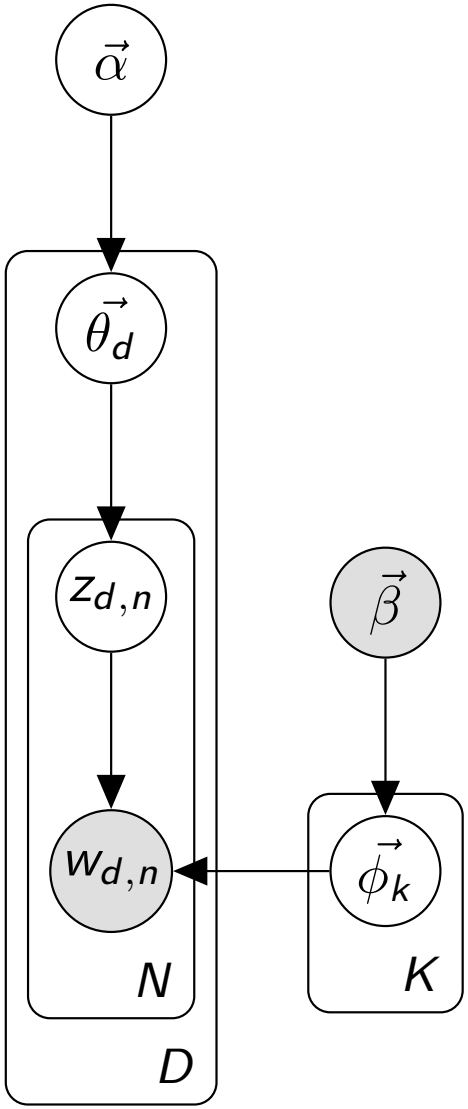
Evolution of Models



LDA- Scalar

original LDA

Evolution of Models, cont.



LDA- Vector

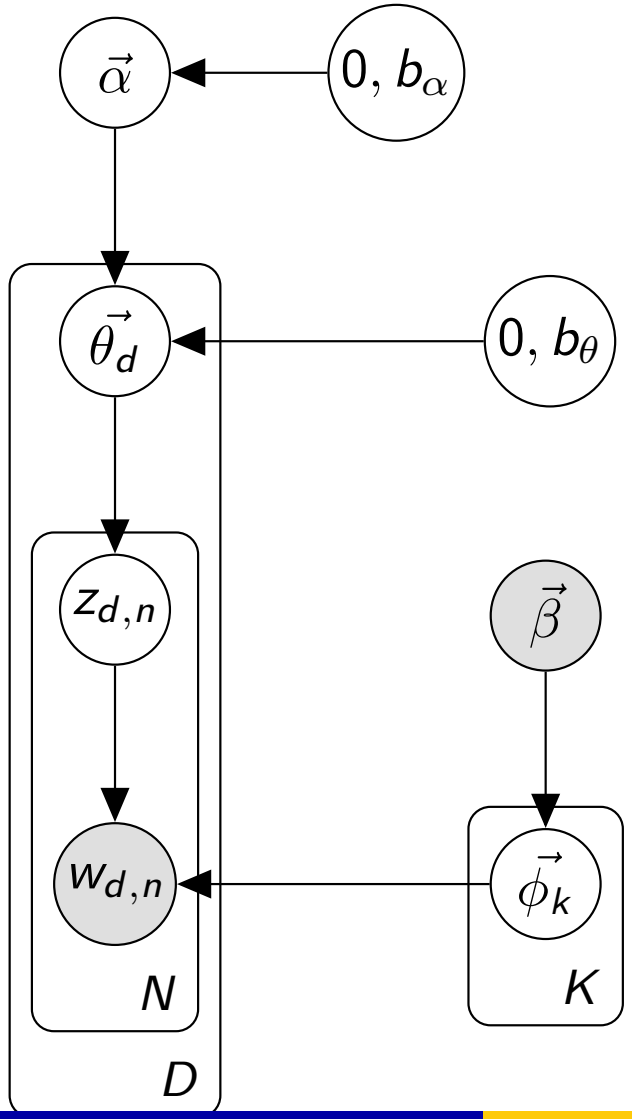
adds asymmetric Dirichlet prior like Wallach et al.;

is also truncated HDP-LDA;

implemented by Mallet since 2008 as assymetric-symmetric LDA

no one knew!

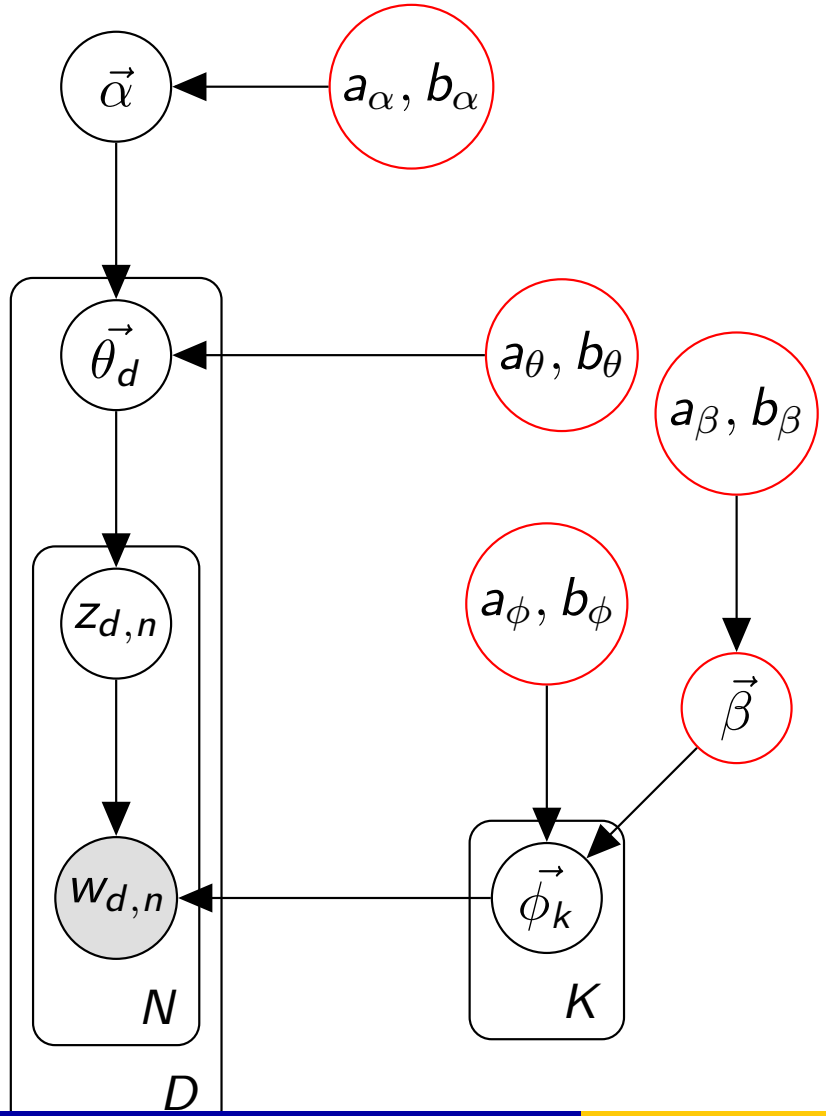
Evolution of Models, cont.



HDP-LDA

adds proper modelling of topic prior like Teh et al.

Evolution of Models, cont.



NP-LDA

adds power law on word distributions like Sato et al. and estimation of background word distribution

Text and Burstiness

Original news article:

Women may only account for 11% of all Lok-Sabha MPs but they fared better when it came to representation in the Cabinet. Six women were sworn in as senior ministers on Monday, accounting for 25% of the Cabinet. They include Swaraj, Gandhi, Najma, Badal, Uma and Smriti.

Bag of words:

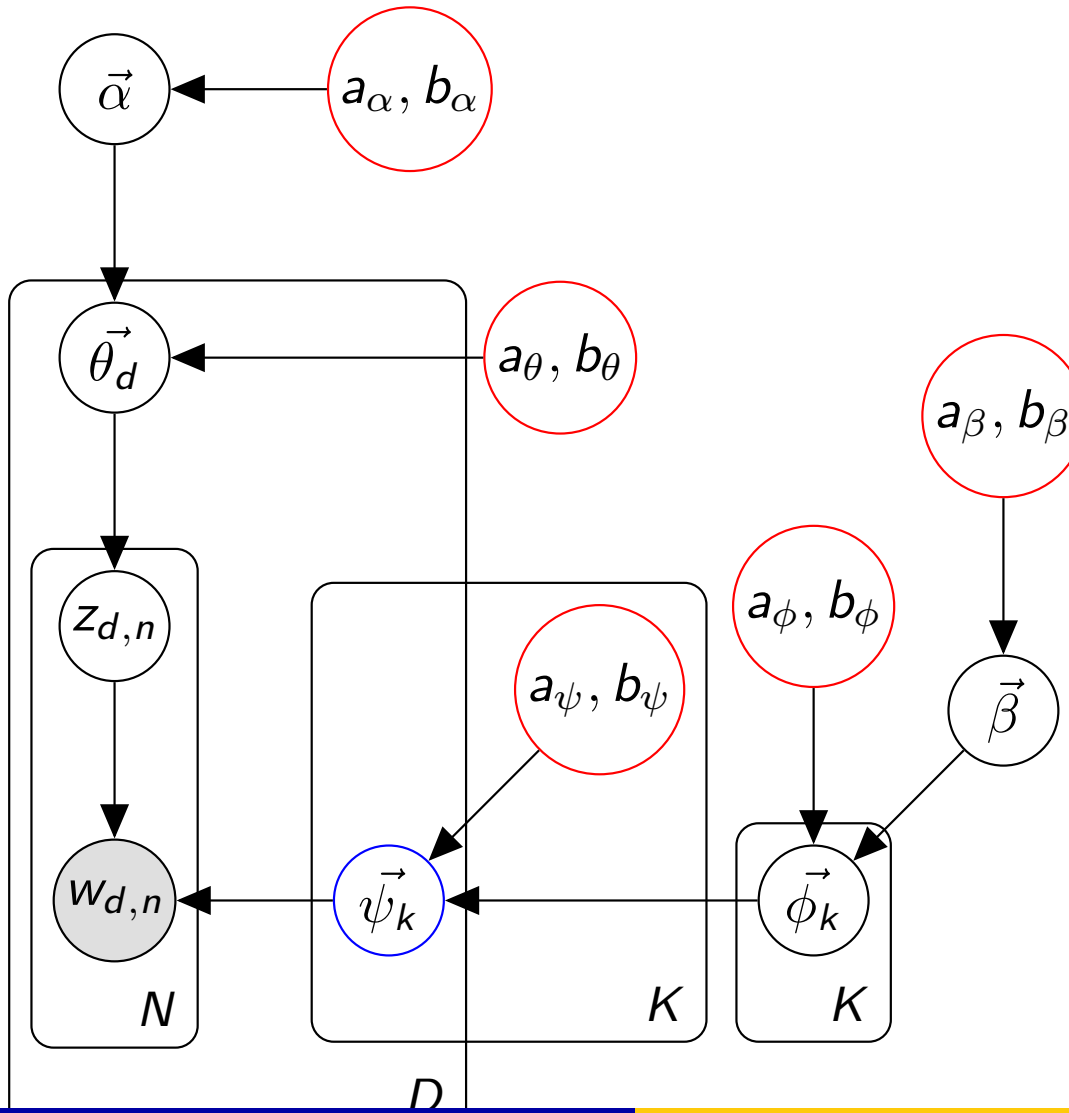
11% 25% Badal Cabinet(2) Gandhi Lok-Sabha MPs Monday Najma Six Smriti Swaraj They Uma Women account accounting all and as better but came fared for(2) in(2) include it may ministers of on only representation senior sworn the(2) they to were when women

NB. "Cabinet" appears twice! It is **bursty**
(see Doyle and Elkan, 2009)

Aside: Burstiness and Information Retrieval

- burstiness and eliteness are concepts in information retrieval used to develop BM25 (*i.e.* dominant TF-IDF version)
- the two-Poisson model and the Pitman-Yor model can be used to justify theory (Sunehag, 2007; Puurula, 2013)
- relationships not yet fully developed

Our Non-parametric Topic Model



$\{\vec{\theta}_d\}$ = document⊗topic matrix

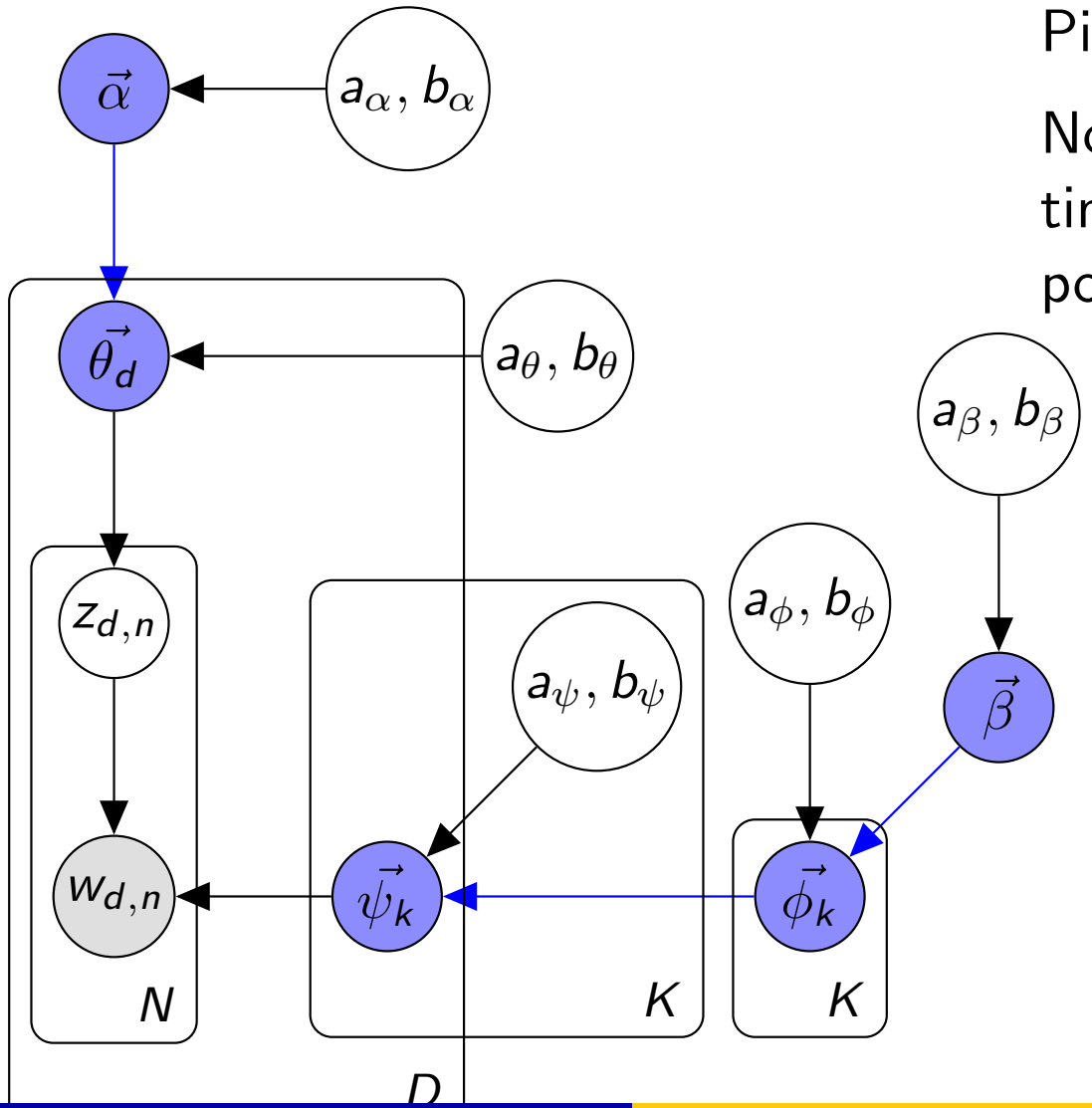
$\{\vec{\phi}_k\}$ = topic⊗word matrix

$\vec{\alpha}$ = prior for document⊗topic matrix

$\vec{\beta}$ = prior for topic⊗word matrix

- Full fitting of priors, and their hyperparameters.
- Topic⊗word vectors $\vec{\phi}_k$ specialised to the document to yield $\vec{\psi}_k$.

Our Non-parametric Topic Model, cont.



The blue nodes+arcs are Pitman-Yor process hierarchies.

Note in $\{\vec{\psi}_k\}$ there are hundreds times more parameters than data points!

Our Non-parametric Topic Model, cont.

Read off the marginalised posterior as follows:

$$\frac{\text{Beta}\left(\vec{n}^{\alpha} + \alpha_\theta \vec{1}/K\right)}{\text{Beta}\left(\alpha_\theta \vec{1}/K\right)} \prod_{d=1}^{D} F_{\theta_\mu}(\vec{n_d}^{\mu}, \vec{t_d}^{\mu}) \qquad \text{(document side)}$$

$$F_{d_\beta,\theta_\beta}(\vec{n}^{\beta}, \vec{t}^{\beta}) \prod_{k=1}^{K} F_{d_\psi,\theta_\psi}(\vec{n_k}^{\psi}, \vec{t_k}^{\psi}) F_{d_\phi,\theta_\phi}(\vec{n_k}^{\phi}, \vec{t_k}^{\phi}) \qquad \text{(word side)}$$

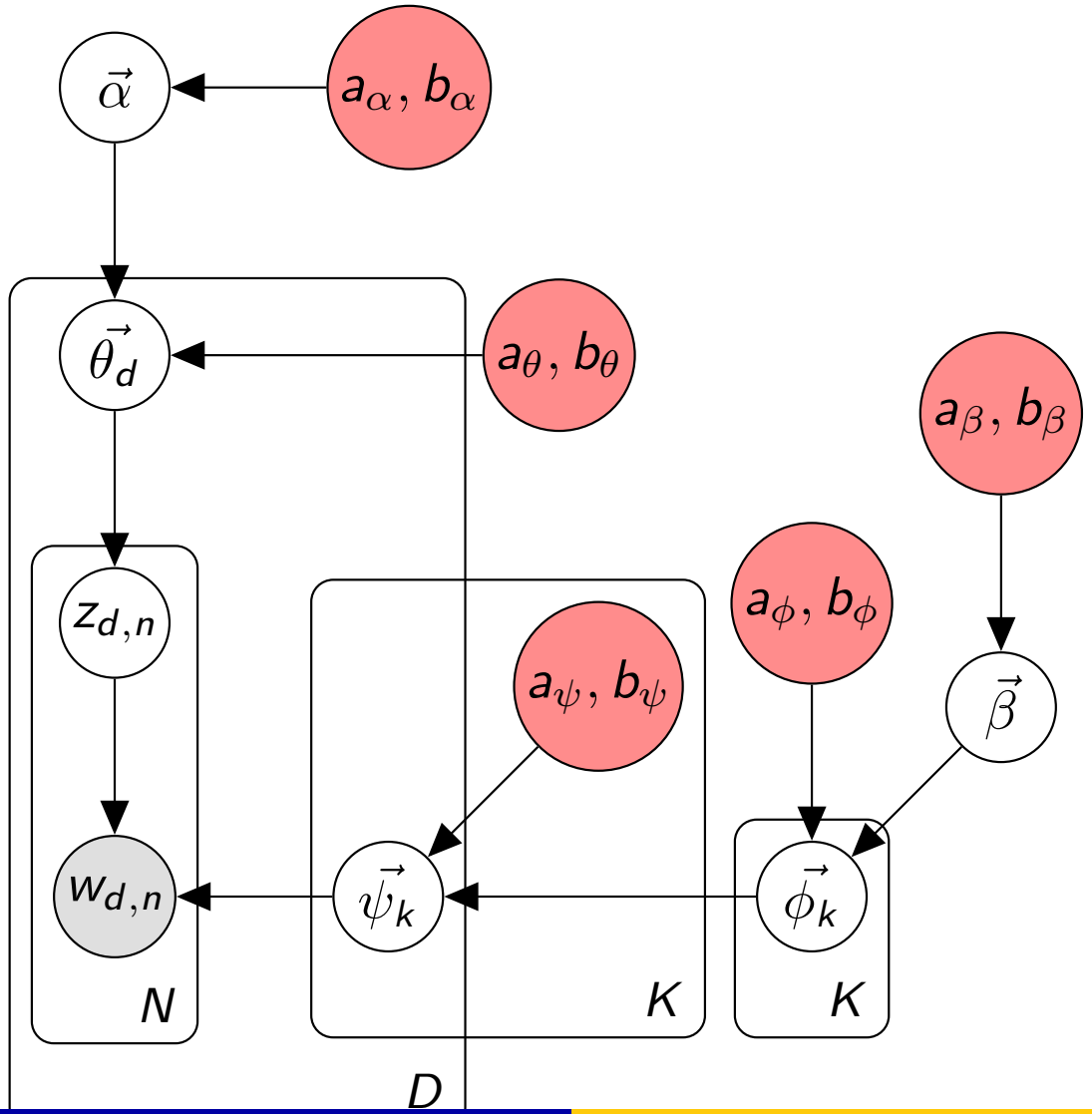
where

$$\vec{n}^{\alpha} = \sum_{d=1}^{D} \vec{t_d}^{\mu} \ , \qquad \vec{n_k}^{\phi} = \vec{t_k}^{\psi} \ , \qquad \vec{n}^{\beta} = \sum_{k=1}^{K} \vec{t_k}^{\phi} \ ,$$

plus all the constraints hold, such as

$$\forall_{k,w} \left(n^{\phi}_{k,w} \geq t^{\phi}_{k,w} \ \ \& \ \ n^{\phi}_{k,w} > 0 \text{ iff } t^{\phi}_{k,w} > 0 \right)$$

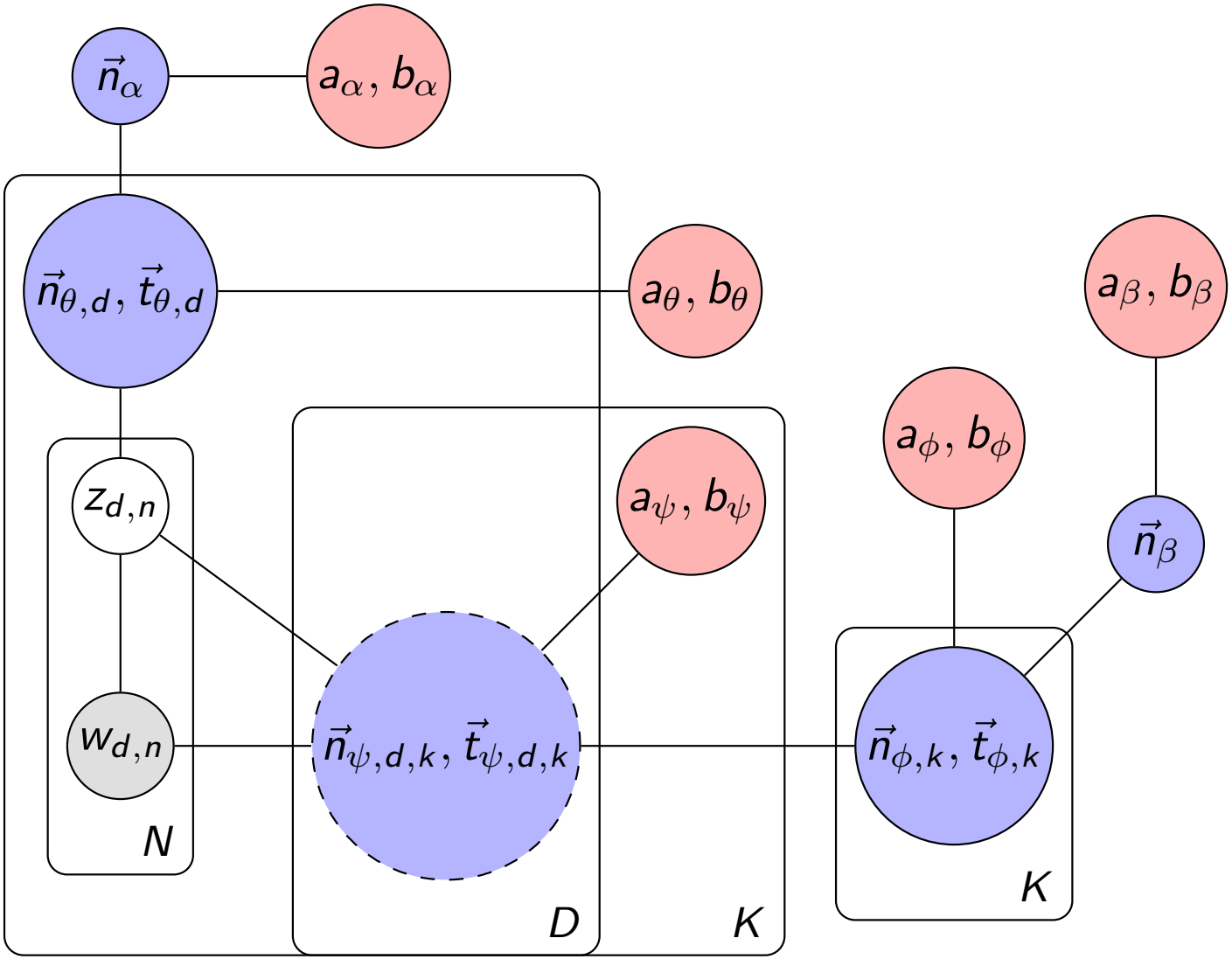
Our Non-parametric Topic Model, cont.



The red nodes are hyper-parameters fit with Adaptive-Rejection sampling or slice sampling.

Use DP on document side ($a_\alpha = 0, a_\theta = 0$) as fitting usually wants this anyway.

Our Non-parametric Topic Model, cont.



Auxiliary latent variables (the $\vec{t}$) propagate part of the counts (their $\vec{n}$) up to the parent.

We keep/recompute sufficient statistics for matrices.

e.g. the $\vec{\psi}$ statistics $\vec{n}_{\psi,d,k}, \vec{t}_{\psi,d,k}$ are not stored but recomputed from booleans as needed.

Double the memory of regular LDA, and only static memory.

Summary: What You Need to Know

PYPs on discrete domains: samples from the SSM get duplicates

PItan-Yor-Multinomial: the PYP variant of the Dirichlet-Multinomial

N-grams: simple example of a hierarchical PYP/DP model

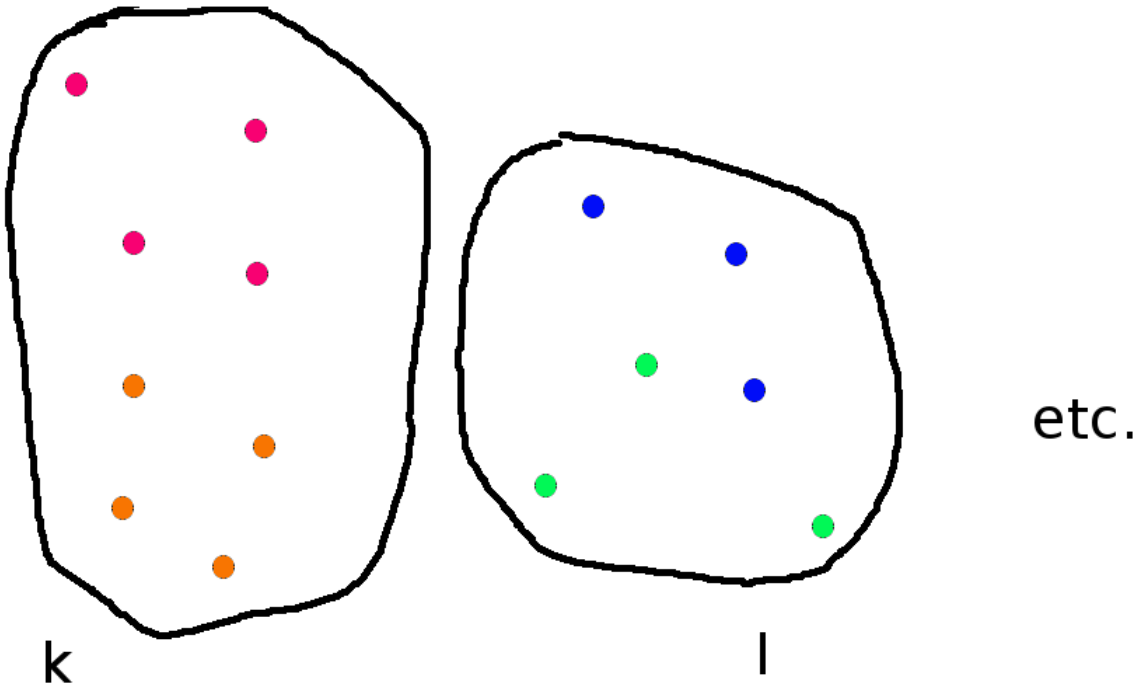
Hierarchical PYPs: the hierarchy of probability vectors are marginalised out leaving a hierarchy of number of tables vectors corresponding to the count vectors.

Structured topic model: STM is a simple extension to LDA showing hierarchical PYPs, see Du, Buntine and Jin (2012)

Simple Gibbs sampling: sampling number of tables vectors individually is poor due to poor mixing.

Outline

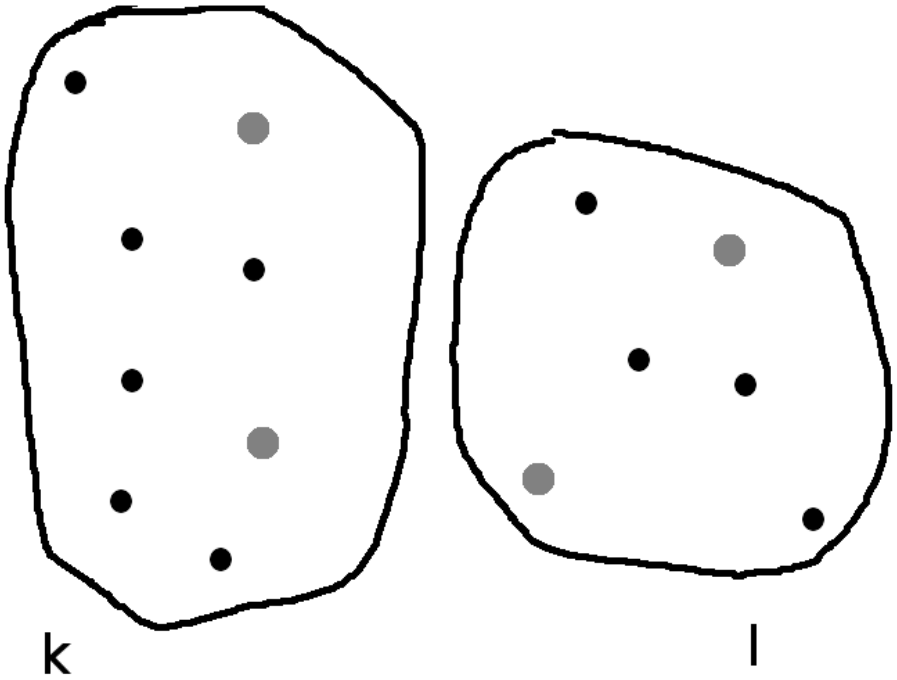
Species with Subspecies



Within species there are separate *sub-species*, pink and orange for type k, blue and green for type l.

Chinese restaurant samplers work in this space, keeping track of all counts for sub-species.

Species with New Species

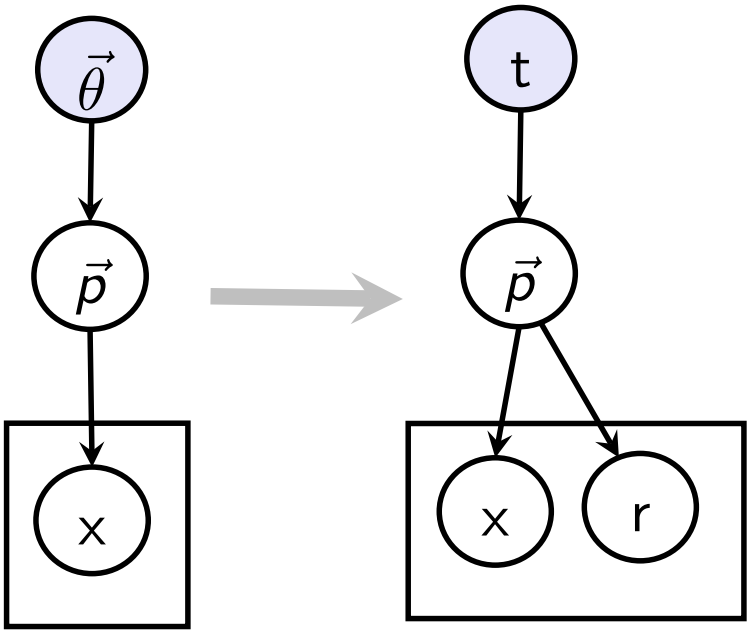


etc.

Within species there are separate *sub-species*, but we only know which data is the first of a new sub-species.

Block table indicator samplers work in this space, where each datum has a Boolean indicator.

Categorical Data plus Table Indicators



LHS = *categorical* form with sample of discrete values $x_1, ..., x_N$ drawn from categorical distribution $\vec{p}$ which in turn has mean $\vec{\theta}$

RHS = *species sampling* form where data is now pairs $(x_1, r_1)..., (x_N, r_N)$ were r_n is a Boolean indicator saying "is new subspecies"

$r_n = 1$ then the sample x_n was drawn from the parent node with probability θ_{x_n}, otherwise is existing subspecies

Table Indicators

Definition of table indicator

Instead of considering the Pitman-Yor-multinomial with counts $(\vec{n}, \vec{t})$, work with sequential data with individual values $(x_1, r_1), (x_2, r_2), ..., (x_N, r_N)$. The table indicator r_n indicates that the data contributes one count up to the parent probability.

So the data is treated sequentially, and taking statistics of $\vec{x}$ and $\vec{r}$ yields:

$$\begin{aligned} n_k &:= \text{counts of } k\text{'s in } \vec{x}, \\ &= \sum_{n=1}^{N} 1_{x_n=k}, \\ t_k &:= \text{counts of } k\text{'s in } \vec{x} \text{ co-occuring with an indicator}, \\ &= \sum_{n=1}^{N} 1_{x_n=k} 1_{r_n}. \end{aligned}$$

The Pitman-Yor-Categorical

Definition of Pitman-Yor-Categorical

Given a concentration parameter α, a discount parameter d, a probability vector $\vec{\theta}$ of dimension L, and a count N, the Pitman-Yor-categorical distribution creates a sequence of discrete class assignments and indicators $(x_1, r_1), ...(x_N, r_N)$. Now $(\vec{x}, \vec{r}) \sim \text{CatPY}\left(d, \alpha, \vec{\theta}, N\right)$ denotes

$$p\left(\vec{x}, \vec{r} \middle| N, \text{CatPY}, d, \alpha, \vec{\theta}\right) = \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{l=1}^{L} \mathcal{S}_{t_l,d}^{n_l} \theta_l^{t_l} \binom{n_l}{t_l}^{-1}$$

where the counts are derived, $t_l = \sum_{n=1}^{N} 1_{x_n=l} 1_{r_l}$, $n_l = \sum_{n=1}^{N} 1_{x_n=l}$, $T = \sum_{l=1}^{L} t_l$.

The Categorical- versus Pitman-Yor-Multinomial

Pitman-Yor-Multinomial: working off counts $\vec{n}, \vec{t}$,

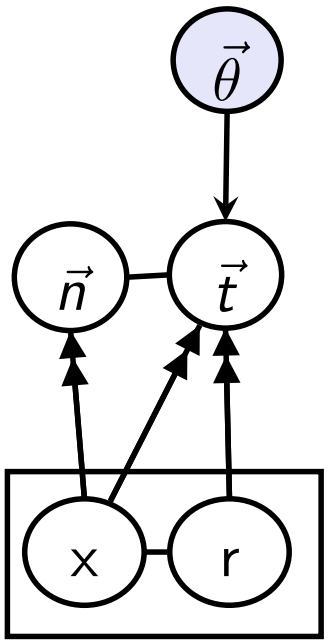
$$p\left(\vec{n}, \vec{t} \middle| N, \text{MultPY}, d, \alpha, \vec{\theta}\right) = \binom{N}{\vec{n}} \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \mathcal{S}_{t_k,d}^{n_k} \theta_k^{t_k}$$

Pitman-Yor-Categorical: working off sequential data $\vec{x}, \vec{r}$, the counts $\vec{n}, \vec{t}$ are now derived,

$$p\left(\vec{x}, \vec{r} \middle| N, \text{CatPY}, d, \alpha, \vec{\theta}\right) = \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \mathcal{S}_{t_k,d}^{n_k} \theta_k^{t_k} \binom{n_k}{t_k}^{-1}$$

- remove the $\binom{N}{\vec{n}}$ term because sequential order now matters
- divide by $\binom{n_k}{t_k}$ because this is the number of ways of distributing the t_k indicators that are on amongst n_k places

Pitman-Yor-Categorical Marginalisation

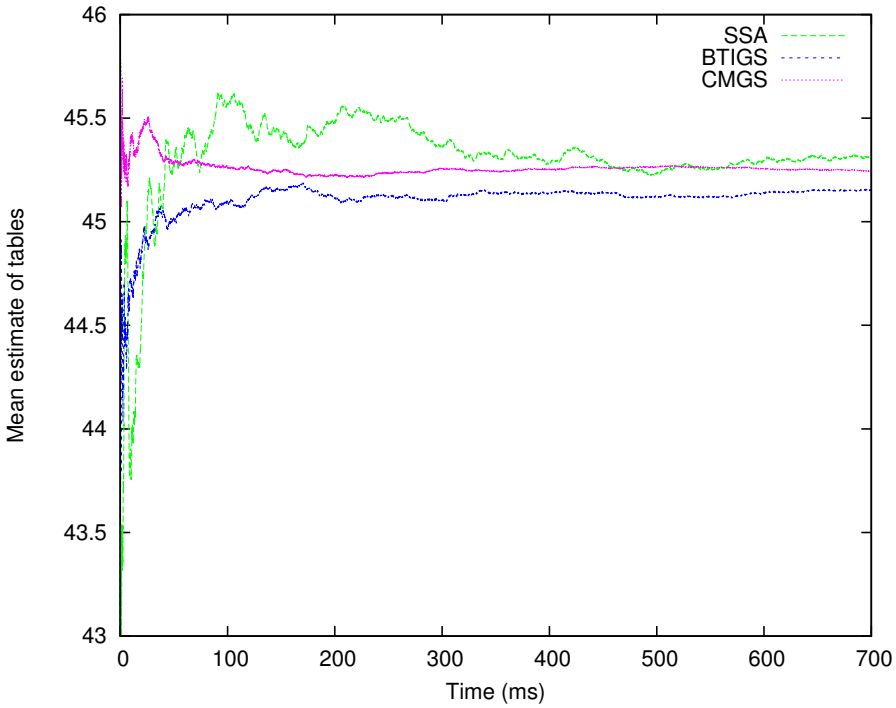


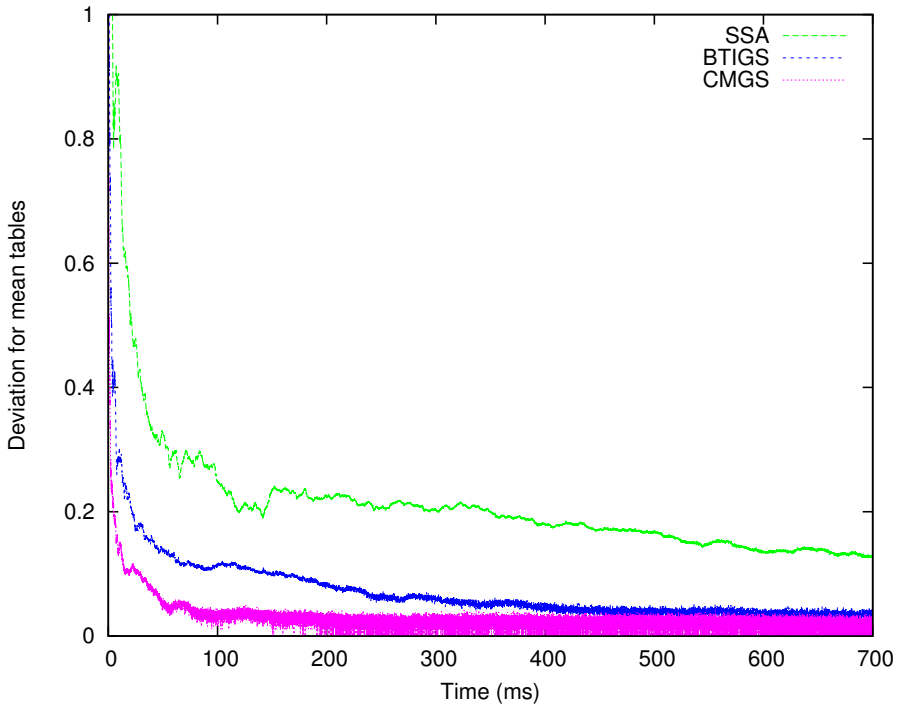
$\vec{n} =$ *vector of counts* of different species (how much data of each species); computed from the data $\vec{x}$

$\vec{t} =$ count vector giving how many different subspecies; computed from the paired data $\vec{x}, \vec{r}$; called *number of tables*

$$p\left(\vec{x}, \vec{r} \middle| d, \alpha, \text{PYP}, \vec{\theta}\right) = \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \theta_k^{t_k} \mathcal{S}_{t_k,d}^{n_k} \binom{n_k}{t_k}^{-1}$$

Comparing Samplers for CatPY versus MultPY

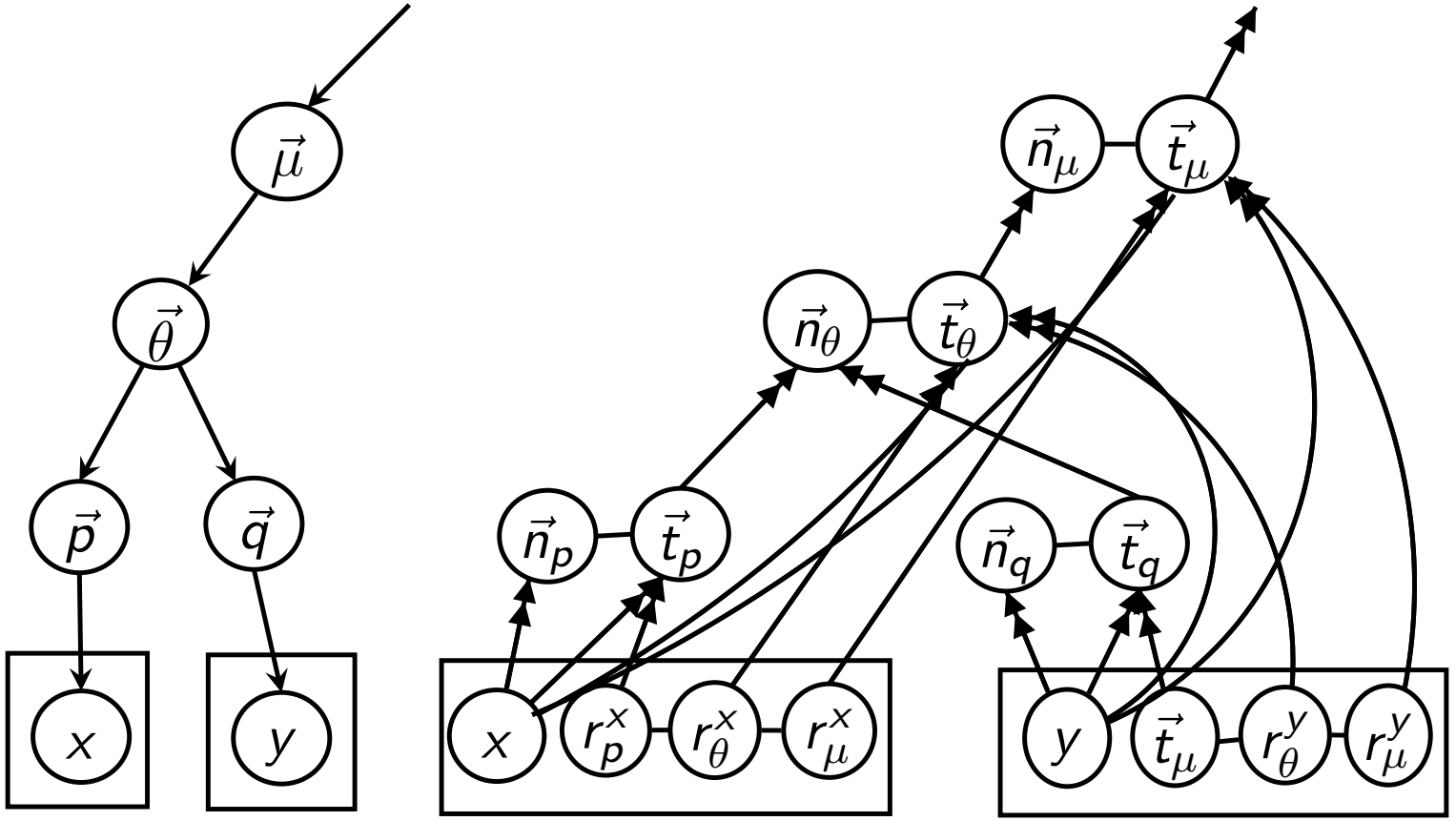




Legend: SSA = "standard CRP sampler of Teh *et al.*"
BTIGS = "Gibbs sampler using CatPY posterior"
CMGS = "Gibbs sampler using MultPY posterior"

Mean estimates of the total number of tables T for one of the 20 Gibbs runs (left) and the standard deviation of the 20 mean estimates (right) with $d = 0$, $\alpha = 10$, $K = 50$ and $N = 500$.

Hierarchical Marginalisation



left is the original probability vector hierarchy, **right** is the result of marginalising out probability vectors then

- indicators are attached to their originating data as a set
- all $\vec{n}$ and $\vec{t}$ counts up the hierarchy are computed from these

Outline

Using the (Categorical) Evidence Formula

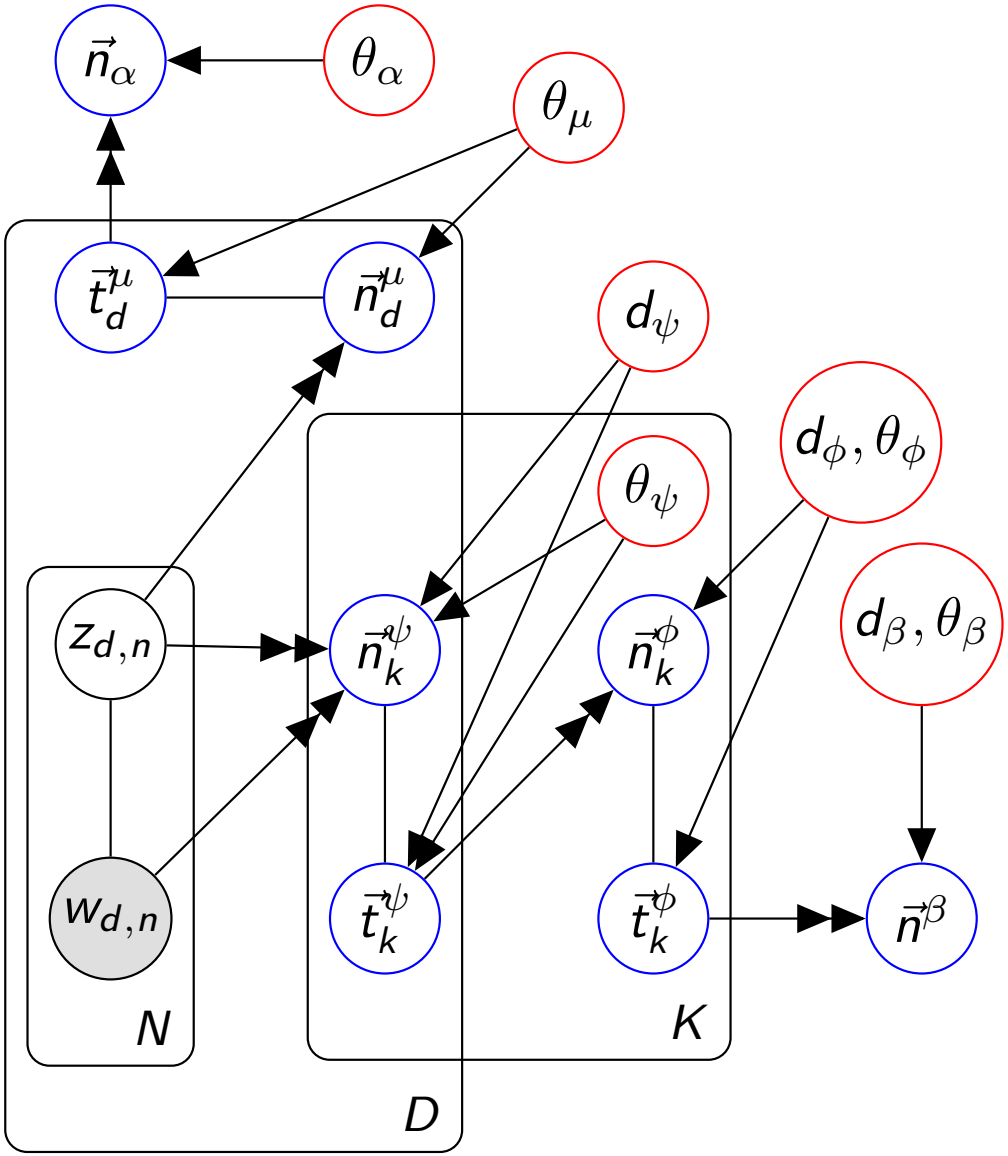
We will repeatedly apply the evidence formula

$$\begin{aligned} p\left(\vec{x}, \vec{t} \,\middle|\, N, \text{CatPY}, d, \alpha\right) &= \frac{(\alpha|d)_T}{(\alpha)_N} \prod_{k=1}^{K} \mathcal{S}^{n_k}_{t_k,d} \binom{n_k}{t_k}^{-1} H(k)^{t_k} \\ &= F'_{d,\alpha}(\vec{n}, \vec{t}) \prod_{k=1}^{K} H(k)^{t_k} \end{aligned}$$

to marginalise out all the probability vectors where

$$F'_{d,\alpha}(\vec{n}, \vec{t}) = F_{d,\alpha}(\vec{n}, \vec{t}) \prod_{k=1}^{K} \binom{n_k}{t_k}^{-1}$$

Marginalised Bursty Non-parametric Topic Model



- started with two hierarchies $\vec{\mu}_d \to \vec{\alpha}$ and $\vec{\psi}_k \to \vec{\phi}_k \to \vec{\beta}$
- counts (in blue) $\vec{n}_d^{\,\mu}$, $\vec{n}^\alpha$, $\vec{n}_k^{\,\psi}$, $\vec{n}_k^{\,\phi}$ and $\vec{n}^\beta$ introduced, and their numbers of tables $\vec{t}_d^{\,\mu}$, *etc.*
- root of each hierarchy modelled with an improper Dirichlet so no $\vec{t}^{\,\alpha}$ or $\vec{t}^{\,\beta}$
- table indicators, not shown, are $r_{d,n}^{\mu}$, $r_{d,n}^{\psi}$, and $r_{d,n}^{\phi}$
- all counts and numbers of tables can be derived from topic $z_{d,n}$ and indicators

Bursty Non-parametric Topic Model, cont.

Modify the evidence to add choose terms to get:

$$E = \frac{\text{Beta}\left(\vec{n}^{\alpha} + \theta_\alpha \vec{1}/K\right)}{\text{Beta}\left(\theta_\alpha \vec{1}/K\right)} \prod_{d=1}^{D} F'_{\theta_\mu}(\vec{n_d}^{\mu}, \vec{t_d}^{\mu}) \qquad \text{(document side)}$$

$$F'_{d_\beta,\theta_\beta}(\vec{n}^{\beta}, \vec{t}^{\beta}) \prod_{k=1}^{K} F'_{d_\psi,\theta_\psi}(\vec{n_k}^{\psi}, \vec{t_k}^{\psi}) F'_{d_\phi,\theta_\phi}(\vec{n_k}^{\phi}, \vec{t_k}^{\phi}) \qquad \text{(word side)}$$

where totals and constraints hold as before, derived as

$$n^{\mu}_{d,k} = \sum_{n=1}^{N} 1_{z_{d,n}=k} \qquad t^{\mu}_{d,k} = \sum_{n=1}^{N} 1_{z_{d,n}=l} 1_{r^{\mu}_{d,n}}$$

$$n^{\psi}_{k,w} = \sum_{n=1}^{N} 1_{z_{d,n}=k} 1_{w_{d,n}=w} \qquad t^{\psi}_{k,w} = \sum_{n=1}^{N} 1_{z_{d,n}=k} 1_{w_{d,n}=w} 1_{r^{\psi}_{d,n}}$$

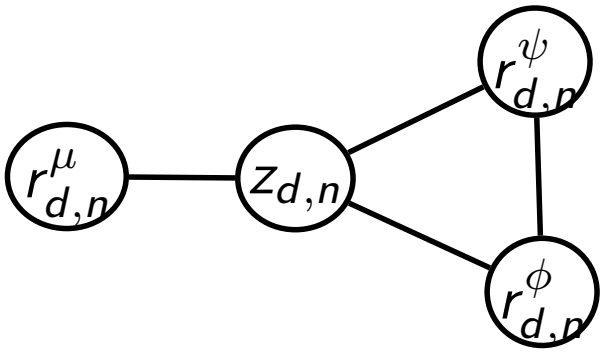
$$n^{\phi}_{k,w} = t^{\psi}_{k,w} \qquad t^{\phi}_{k,w} = \sum_{n=1}^{N} 1_{z_{d,n}=k} 1_{w_{d,n}=w} 1_{r^{\psi}_{d,n}} 1_{r^{\phi}_{d,n}}$$

$$n^{\beta}_{w} = \sum_{k} t^{\phi}_{k,w} \qquad t^{\beta}_{w} = 1_{n^{\beta}_{w}>0}$$

Block Sampler for Bursty Non-parametric Topic Model

At the core of the block Gibbs sample, we need to reestimate $(z_{d,n}, r^{\mu}_{d,n}, r^{\psi}_{d,n}, r^{\phi}_{d,n})$ for all documents d and words n.

Considering the evidence (previous slide) as a function of these, $E(z_{d,n}, r^{\mu}_{d,n}, r^{\psi}_{d,n}, r^{\phi}_{d,n})$, we get the graphical model below left:



With belief propagation algorithms, it is easy to:

- compute the marginal contribution for $z_{d,n}$,

$$\sum_{r^{\mu}_{d,n}, r^{\psi}_{d,n}, r^{\phi}_{d,n}} E(z_{d,n}, r^{\mu}_{d,n}, r^{\psi}_{d,n}, r^{\phi}_{d,n})$$

  needed for a block Gibbs sampler
- sample $(r^{\mu}_{d,n}, r^{\psi}_{d,n}, r^{\phi}_{d,n})$ for given $z_{d,n}$

LDA Versus NP-LDA Samplers

	LDA	NP-LDA with table indicators
latent vars	word topics $\vec{z}$	word topics $\vec{z}$, and boolean table indicators $r^{\vec{\mu}}$, $r^{\vec{\psi}}$, $r^{\vec{\phi}}$
derived vectors	topic count $\vec{n}_d^{\mu}$ and word count $\vec{n}_k^{\phi}$	topic count $\vec{n}_d^{\mu}, \vec{t}_d^{\mu}, \vec{n}^{\alpha}$ word count $\vec{n}_k^{\phi}, \vec{n}_k^{\psi}, \vec{t}_k^{\psi}, \vec{n}^{\beta}$
totals kept	$\vec{N}_d^{\mu}$, $\vec{N}_k^{\phi}$	$\vec{N}_d^{\mu}$, $\vec{T}_d^{\mu}$, $\vec{N}_k^{\phi}$, $\vec{N}_k^{\psi}$, $\vec{T}_k^{\psi}$
Gibbs method	on each $z_{d,n}$	blockwise on $(z_{d,n}, r_{d,n}^{\mu}, r_{d,n}^{\psi}, r_{d,n}^{\phi})$

Notes:

- table indicators don't have to be stored but can be resampled as needed by uniform assignment.
- block sampler and posterior form with table indicators are more complex!

Outline

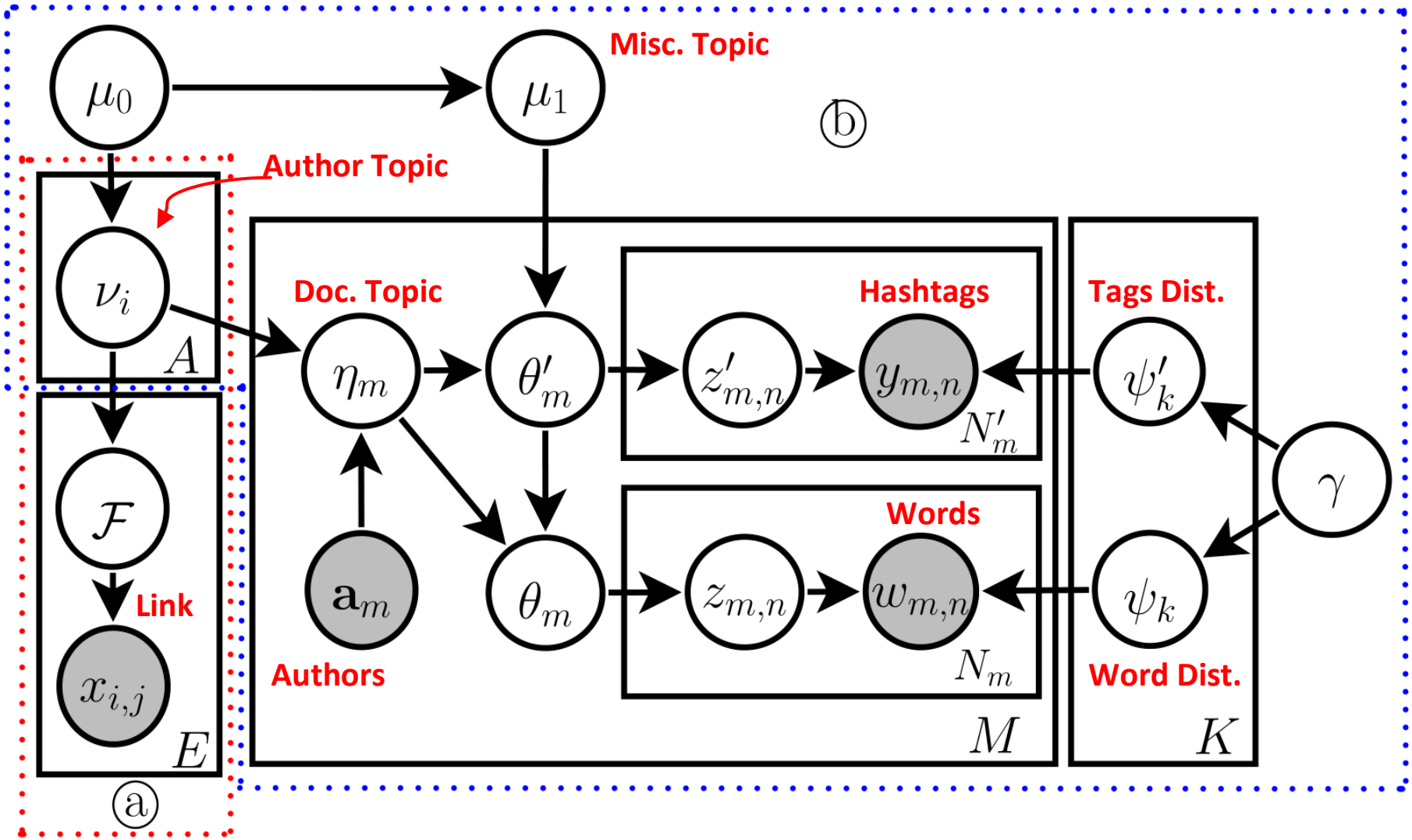
Analysing Tweets

Tweets have a number of facts that make them novel/challenging to study:

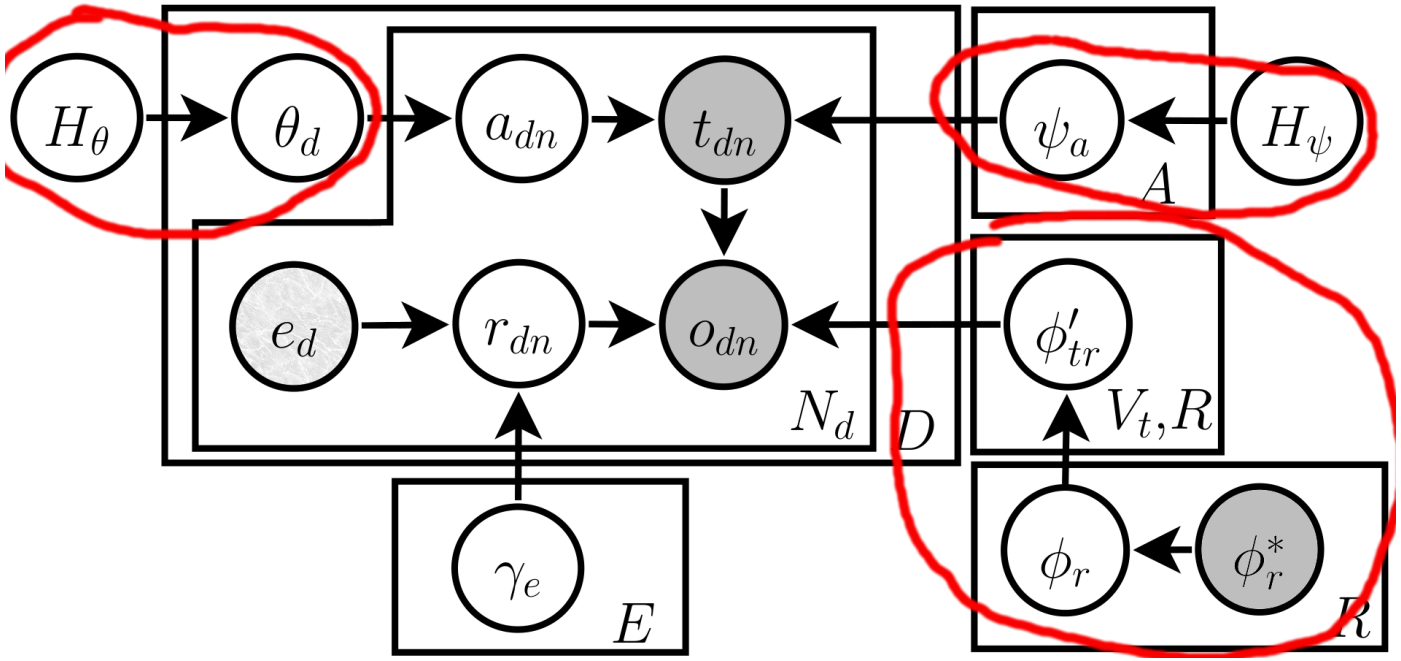
- hashtags, embedded URLs, and retweets,
- small size, informal language and emoticons.
- authors and follower networks,
- frequency in time.

Twitter-Network Topic Model

Kar Wai Lim *et al.*, 2014, submitted



Twitter Opinion Topic Model



"Twitter Opinion Topic Model: Extracting Product Opinions from Tweets by Leveraging Hashtags and Sentiment Lexicon," Lim and Buntine, CIKM 2014

(probability vector hierarchies circled in red)

Unsupervised Part of Speech

A Hierarchical Pitman-Yor Process HMM for Unsupervised Part of Speech Induction

Phil Blunsom
Department of Computer Science
University of Oxford

Trevor Cohn
Department of Computer Science
University of Sheffield

We develop a trigram hidden Markov model which models the joint probability of a sequence of latent tags, $\mathbf{t}$, and words, $\mathbf{w}$, as

$$P_\theta(\mathbf{t}, \mathbf{w}) = \prod_{l=1}^{L+1} P_\theta(t_l|t_{l-1}, t_{l-2}) P_\theta(w_l|t_l) \,,$$

Unsupervised Part of Speech, cont.

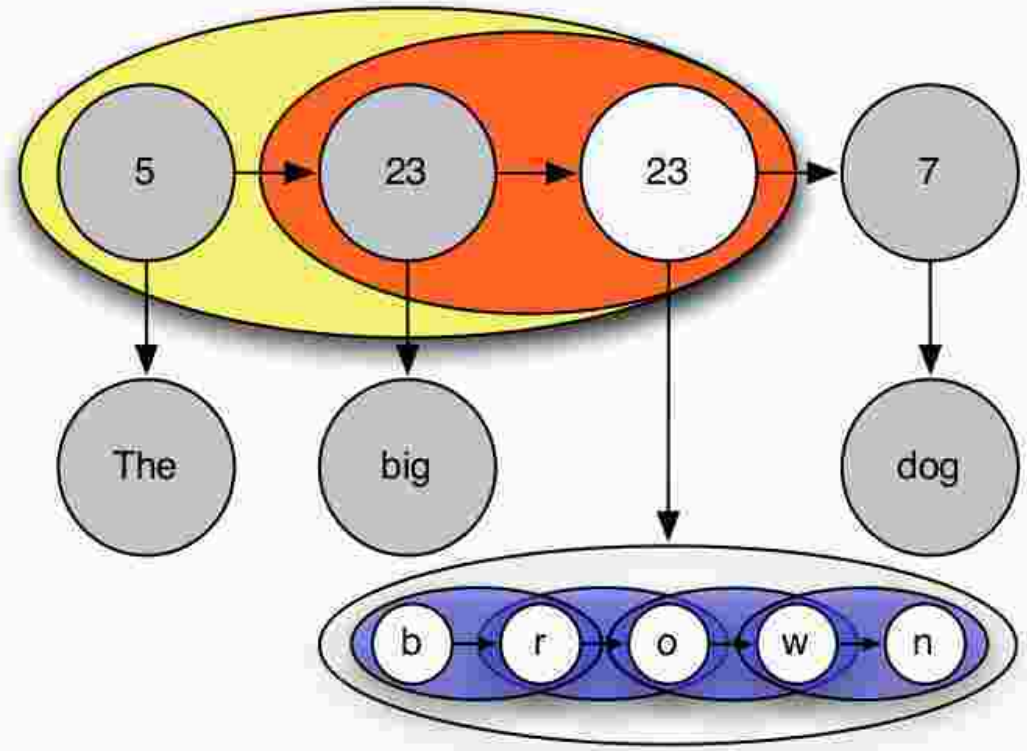


Figure 2: The conditioning structure of the hierarchical PYP with an embedded character language models.

Unsupervised Part of Speech, cont.

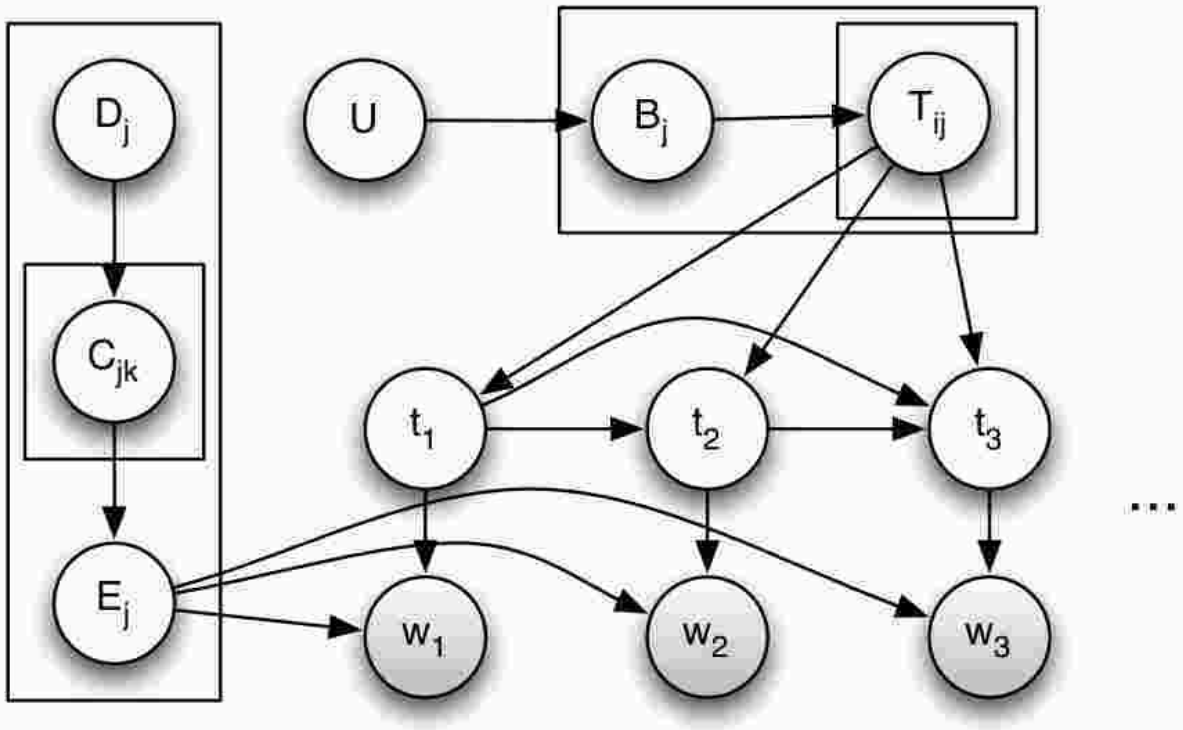
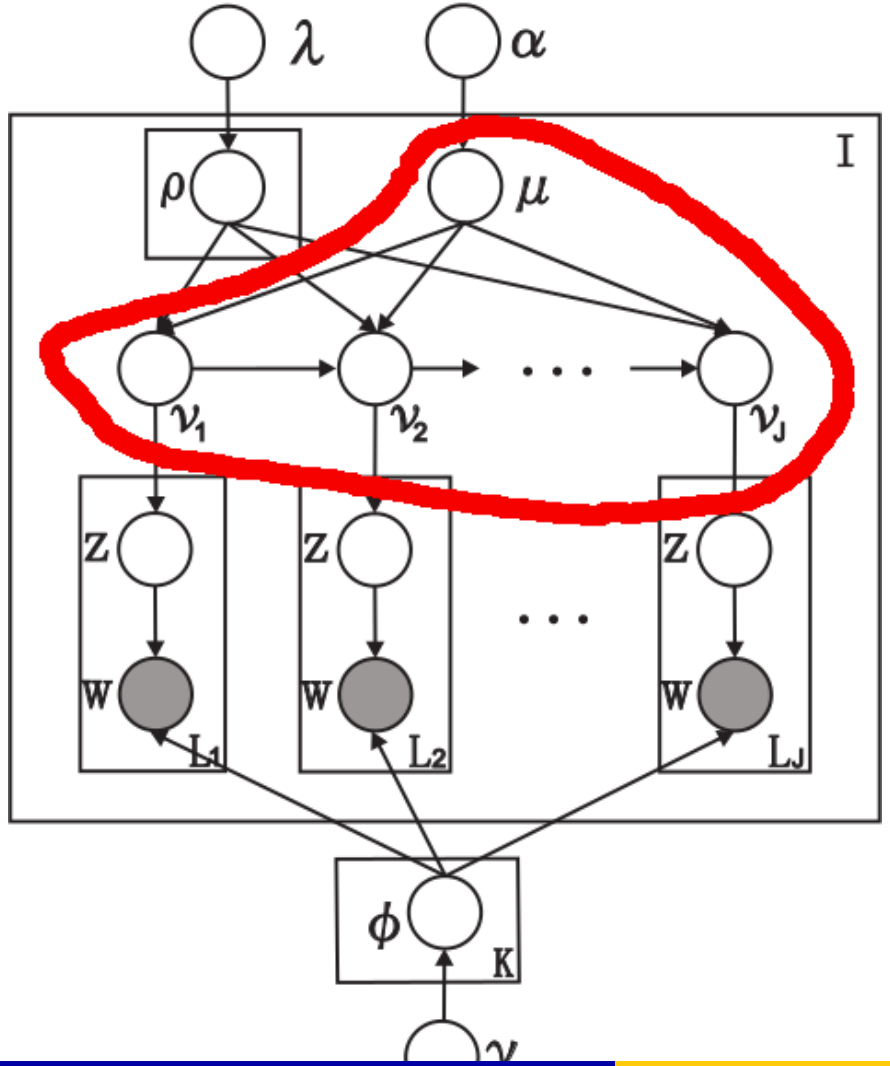


Figure 1: Plate diagram representation of the trigram HMM. The indexes i and j range over the set of tags and k ranges over the set of characters. Hyper-parameters have been omitted from the figure for clarity.

Adaptive Sequential Topic Model

A more complex (sequential) document model.



- The PYPs exist in long chains $(\vec{\nu}_1, \vec{\nu}_2, \ldots, \vec{\nu}_J)$.
- A single probability vector $\vec{\nu}_j$ can have two parents, $\vec{\nu}_{j-1}$ and $\vec{\mu}$.
- More complex chains of table indicators and block sampling.
- See Du *et al.*, EMNLP 2012.

Dynamic Topic Model

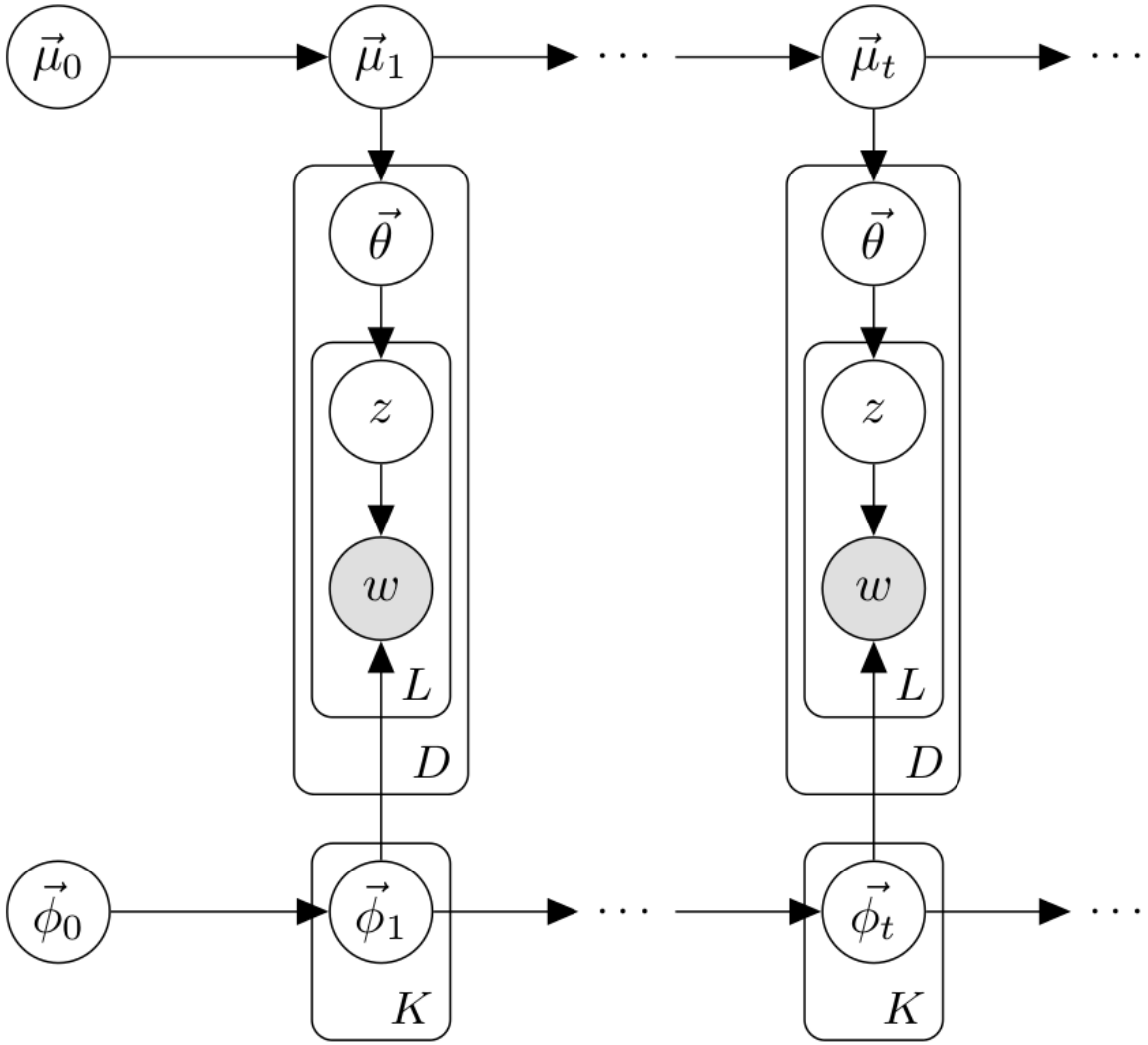
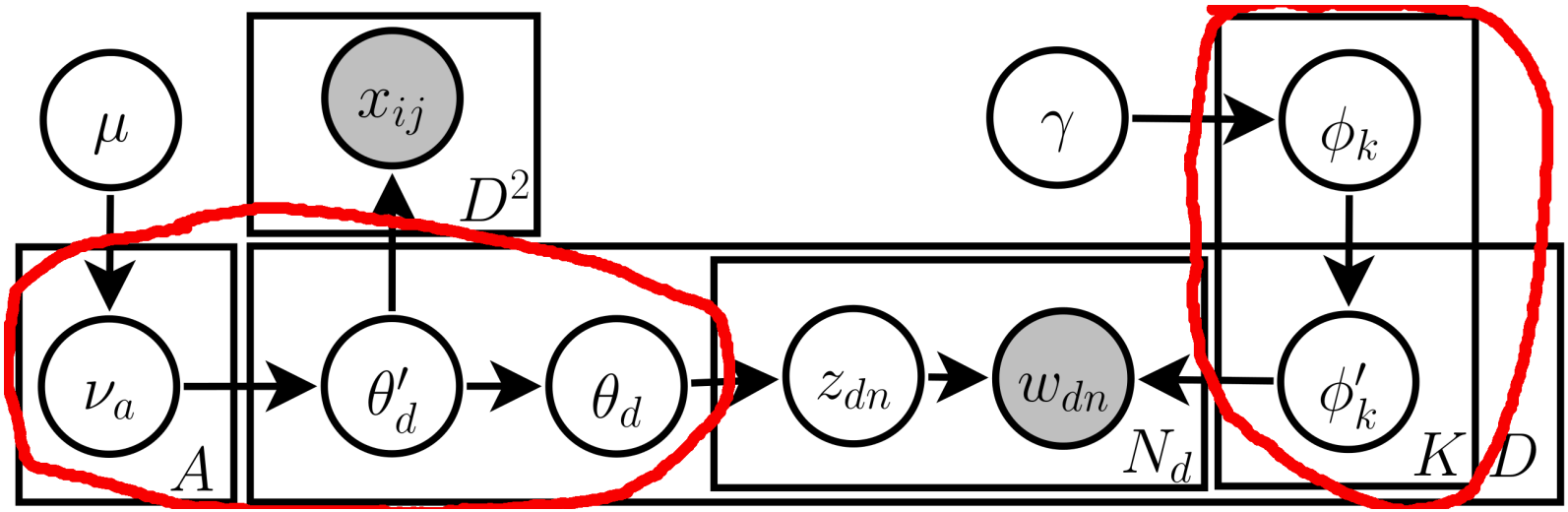


Figure 1: Graphical representation of the proposed model (for epoch 1 and t)

- model is a *sequence* of LDA style topic models chained together
- block table indicator sampling uses caching to work efficiently

Author-Citation Topic Model



"Bibliographic Analysis with the Citation Network Topic Model," Lim and Buntine, ACML, 2014

(probability vector hierarchies circled in red)

Other Regular Extended Models

All of these other related models can be made non-parametric using probability network hierarchies.

Stochastic block models: finding community structure in networks; mixed membership models; bi-clustering;

Infinite hidden relational models: tensor/multi-table extension of stochastic block models;

Tensor component models: tensor extension of component models;

Event clustering: events can be represented in a semi-structured way; how do we develop split and merge of event groupings over time

Outline

Latent Semantic Modelling

- Variety of component and network models in NLP and social networks can be made non-parametric with deep probability vector networks.
- New fast methods for training deep probability vector networks.
- Allows modelling of latent semantics:
  - semantic resources to integrate, (WordNet, sentiment dictionaries, *etc.*),
  - inheritance and shared learning across multiple instances,
  - hierarchical modelling,
  - deep latent semantics,
  - integrating semi-structured and networked content,

i.e. Same as deep neural networks!

Outline

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